

Maximal varieties and the local Langlands correspondence for $GL(n)$

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Abstract

In the second author's study of deformation spaces of formal modules, there arose a certain variety X , defined over a finite field, which was conjectured to have the property of “maximality”: the number of rational points of X is the largest possible among varieties with the same Betti numbers as X . In the current paper we prove this conjecture, and indeed give a complete description of the zeta function of X . As a consequence, the cohomology of X is shown to realize a piece of the local Langlands correspondence for certain wild Weil parameters of low conductor.

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1 Introduction

Let K be a nonarchimedean local field with ring of integers \mathcal{O}_K and residue field k , and let $n \geq 1$ be an integer. The local Langlands correspondence and the Jacquet-Langlands correspondence for $\mathrm{GL}_n(K)$ are both realized in the cohomology of the Lubin-Tate tower $\mathcal{M}^n = \varprojlim \mathcal{M}_m^n$, where \mathcal{M}_m^n is the rigid analytic space parameterizing deformations of a fixed one-dimensional formal \mathcal{O}_K -module of height n over \bar{k} together with a Drinfeld level m structure. (For the precise statement, see the introduction to [HT01].) At present, this fact can only be proved using global methods. A program initiated by the second author in [Wei10] aims to obtain a purely local proof by constructing a sufficiently nice integral model of \mathcal{M}^n and computing $H_c^*(\mathcal{M}^n, \overline{\mathbb{Q}}_\ell)$ as the cohomology of the nearby cycles complex on the special fiber of this model. Here ℓ is a fixed prime different from $p := \mathrm{char} k$.

This idea has roots in the work of T. Yoshida [Yos10], who found an open affinoid in \mathcal{M}_1^n whose reduction turned out to be a certain Deligne-Lusztig variety for the group GL_n over k . Using this affinoid, Yoshida showed by purely local methods that the local Langlands correspondence for *depth zero* supercuspidal representations of $\mathrm{GL}_n(K)$ is realized in the cohomology of \mathcal{M}_1^n . In [Wei10], the second author found an open affinoid in a space intermediate between \mathcal{M}_1^n and \mathcal{M}_2^n whose reduction, a nonsingular variety X defined over the degree n extension of k , is the focus of this article. (There it was assumed that K has positive characteristic, but the same results are expected to hold in the mixed characteristic case.) In that paper it was conjectured (Conj. 1.6) that a piece of the zeta function of X takes a strikingly simple form (see Thm. 3.2). This conjecture implies that the cohomology of X realizes the local Langlands correspondence for a certain class of supercuspidals of $\mathrm{GL}_n(K)$ of positive depth (but with small conductor, see §2.2).

In this article we give a complete description of the cohomology of X in all degrees, thus proving Conj. 1.6 of [Wei10]. We list here some salient properties of the variety X and refer to Section 4 for more details.

- *Analogy with Deligne-Lusztig theory.* The variety X admits an action of the group of rational points of a certain unipotent group U over a finite field¹ \mathbf{F}_Q . In fact, X is the preimage under

¹With the earlier notation, if q is the cardinality of k , then $Q = q^n$.

the Lang map $x \mapsto \text{Fr}_Q(x)x^{-1}$ of a certain subvariety $Y \subset U$ and $U(\mathbf{F}_Q)$ acts on X by right multiplication. (Here Fr_Q is the Q th power Frobenius map.) One can give a simple characterization of the irreducible representations of $U(\mathbf{F}_Q)$ which appear in $H_c^*(X, \overline{\mathbb{Q}}_\ell)$ (see Thm. 4.4). Each such irreducible representation appears with multiplicity one. In all these respects X is analogous to certain varieties that arise in the Deligne-Lusztig theory for reductive groups over finite fields (see, e.g., [Car85, §7.2]).

- *Maximality.* Let S be any scheme of finite type over a finite field \mathbf{F}_Q . It follows from [Del80], Thm. 3.3.1, that for each i and every eigenvalue α of Fr_Q acting on $H_c^i(S, \overline{\mathbb{Q}}_\ell)$, there exists an integer $m \leq i$ such that all complex conjugates of α have absolute value $Q^{m/2}$. So the Grothendieck-Lefschetz trace formula

$$\#S(\mathbf{F}_Q) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}(\text{Fr}_Q, H_c^i(S, \overline{\mathbb{Q}}_\ell))$$

implies the following bound on the number of rational points of S :

$$\#S(\mathbf{F}_Q) \leq \sum_{i \in \mathbb{Z}} Q^{i/2} \dim H_c^i(S, \overline{\mathbb{Q}}_\ell).$$

This bound is achieved if and only if Fr_Q acts on $H_c^i(S, \overline{\mathbb{Q}}_\ell)$ via the scalar $(-1)^i Q^{i/2}$ for each i , in which case the scheme S is called *maximal*. (There are plenty of references in the literature to “maximal curves” over finite fields: these are smooth projective curves which attain the Hasse-Weil bound on the number of rational points. As far as we know, our definition of maximality for arbitrary schemes over a finite field is new.) By Cor. 4.7, X is a maximal variety over \mathbf{F}_Q .

- *Manifestation of the local Langlands correspondence.* In §3.5 we explain how our main theorem (Thm. 4.4) can be used to show that both the local Langlands and Jacquet-Langlands correspondences for a certain class of wild supercuspidals are realized in the cohomology of an open piece of the Lubin-Tate tower. See Thm. 3.6.

1.1 Outline of the paper

In §2.1 we give an overview of the local Langlands correspondence and the Jacquet-Langlands correspondence for $\text{GL}(n)$ over a nonarchimedean local field K . We will only consider the twist D^\times of

$\mathrm{GL}_n(K)$ arising as the multiplicative group of the central division algebra D/K of invariant $1/n$. On the one hand, the correspondence admits a geometric realization in the cohomology of the Lubin-Tate tower. On the other hand, for some Weil parameters it is known how to explicitly construct the corresponding representations of $\mathrm{GL}_n(K)$ and D^\times . This paper focuses on a certain class of irreducible Weil parameters, defined in §2.2, which are wildly ramified, but have small conductor. The corresponding representations of $\mathrm{GL}_n(K)$ and D^\times are constructed explicitly in §2.2 and §2.3. The representation of D^\times is induced from an irreducible character of a certain subquotient of D^\times , whose p -primary part is a finite p -group U .

It so happens that U is the group of \mathbf{F}_{q^n} -points of a connected unipotent algebraic group, referred to in the paper as $U^{n,q}$, defined over the residue field \mathbf{F}_q of K . This algebraic group is the starting point for the definition of the variety X , given in §3.1. X is a non-singular affine variety endowed with a right action of U . The main theorem of the paper is a complete description of the ℓ -adic cohomology of X as a module for both the action of U and for Frobenius. However, the theorem is much simpler to state in the so-called primitive case, which we announce in §3.3. An application of this case of the theorem is given in §3.5. We define an action on X by a subgroup of $\mathrm{GL}_n(K) \times D^\times \times W_K$ (which has U as a subquotient). Then we show that the representation of $\mathrm{GL}_n(K) \times D^\times \times W_K$ induced from the middle cohomology of X realizes the local Langlands and Jacquet-Langlands correspondences for exactly the class of Weil parameters under consideration.

In §4, the general case of the main theorem (Thm. 4.4) is announced. It states that for every character ψ of the center of U , there is a unique irreducible representation ρ_ψ of U which lies over ψ and which appears in the cohomology of X . Furthermore, each ρ_ψ appears exactly once. Both the degree of the cohomology in which ρ_ψ appears and the action of Frobenius on it are made explicit in terms of ψ . As a consequence it is shown that X is a maximal variety (Cor. 4.7).

The remainder of the paper is devoted to the proof of the main theorem. In §5 we establish some generalities regarding ℓ -adic local systems on algebraic groups G defined over a finite field \mathbf{F}_q . Of particular importance is the Lang isogeny, cf. 5.5, which can be used to associate a local system \mathcal{E}_ρ to a \mathbb{Q}_ℓ -valued representation ρ of $G(\mathbf{F}_q)$. When ρ is induced from a representation η of $H(\mathbf{F}_q)$ for a connected subgroup $H \subset G$, we derive an important relationship (Prop. 5.16)

between \mathcal{E}_ρ and \mathcal{E}_η .

The final two sections of the paper specialize to the case of our group $U^{n,q}$. §6 establishes some properties of the “reduced norm map” $U^{n,q} \rightarrow \mathbb{G}_a$, which is a geometric version of the reduced norm map $D^\times \rightarrow F^\times$. The heart of the proof of the main theorem is in §7. A step-by-step outline of the argument can be found in §7.1.

Since the proof of Thm. 4.4 is somewhat long and complicated, let us summarize the key underlying ideas. First, X is defined as the preimage of a certain hyperplane $Y \subset U^{n,q}$ under the Lang map $L_Q : U^{n,q} \rightarrow U^{n,q}$, $g \mapsto \text{Fr}_Q(g) \cdot g^{-1}$, and the action of $U = U^{n,q}(\mathbf{F}_Q)$ on X comes from the right multiplication action of U on $U^{n,q}$. It is not hard to show that for every representation ρ of U over $\overline{\mathbb{Q}}_\ell$, the “ ρ -isotypic component” of $H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)$ can be naturally identified with $H_c^\bullet(Y, \mathcal{E}_\rho)$, where \mathcal{E}_ρ is the local system on $U^{n,q}$ associated to ρ (see Cor. 5.13). One then reduces the proof of Thm. 4.4 to the calculation of $H_c^\bullet(Y, \mathcal{E}_\rho)$ for certain representations ρ that can be induced from 1-dimensional representations of groups of \mathbf{F}_{q^n} -points of connected subgroups of $U^{n,q}$. Using Prop. 5.16 one identifies $H_c^\bullet(Y, \mathcal{E}_\rho)$ with the cohomology of certain rank 1 local systems on affine spaces. Finally, the latter turn out to be amenable to an inductive calculation using certain linear fibrations of affine spaces $\mathbb{A}^d \rightarrow \mathbb{A}^{d-1}$, the proper base change theorem (Thm. 5.4) and the projection formula (Thm. 5.5).

2 Non-abelian Lubin-Tate theory

In this section we establish notation and recall basic facts concerning the local Langlands correspondence, the Jacquet-Langlands correspondence, and the geometric realization of both. We also define a class of wildly ramified irreducible Weil parameters on which we make both correspondences completely explicit.

2.1 The local Langlands correspondence for $\text{GL}(n)$

Recall that K is a nonarchimedean local field of residue characteristic p .

Theorem 2.1 ([LRS93],[Hen00],[HT01]). *There is a bijection $\sigma \mapsto \pi(\sigma)$, $\pi \mapsto \sigma(\pi)$ (the local Langlands correspondence) between the following two sets:*

1. The set $\mathcal{G}_n(K)_{\mathbb{C}}$ of isomorphism classes of complex Frobenius-semisimple Weil-Deligne representations σ of K (“Weil parameters”)
2. The set $\mathcal{A}_n(K)_{\mathbb{C}}$ of isomorphism classes of complex irreducible admissible representations π of $\mathrm{GL}_n(K)$.

The bijection preserves L - and ε -factors of pairs, and is compatible with local class field theory and twisting by one-dimensional characters. Furthermore, $\sigma \mapsto \pi(\sigma)$ puts irreducible Weil representations into bijection with supercuspidal representations.

Let $\sigma^{\#}(\pi) = \sigma(\pi) \left(\frac{1-n}{2} \right)$. Then $\pi \mapsto \sigma^{\#}(\pi)$ is a different bijection $\mathcal{A}_n(K)_{\mathbb{C}} \rightarrow \mathcal{G}_n(K)_{\mathbb{C}}$ which has the advantage of being invariant under automorphisms of the coefficient field. As a result, if $\iota: E \rightarrow \mathbb{C}$ is an isomorphism from any field onto \mathbb{C} , then $\sigma^{\#}$ induces a well-defined bijection $\mathcal{A}_n(K)_E \rightarrow \mathcal{G}_n(K)_E$ which we also notate as $\pi \mapsto \sigma^{\#}(\pi)$.

The local Langlands correspondence for irreducible Weil parameters is manifested geometrically in the cohomology of the Lubin-Tate tower. Indeed, this is how the existence of the correspondence for p -adic K is established in [HT01]. The Lubin-Tate tower $\mathcal{M}^n = \varprojlim \mathcal{M}_m^n$ is the deformation space with level structure of a fixed one-dimensional formal \mathcal{O}_K -module Σ . Let $D = \mathrm{End}(\Sigma) \otimes_{\mathcal{O}_K} K$, so that D/K is the central division algebra of invariant $1/n$. Then \mathcal{M}^n admits an action of a subgroup $H \subset \mathrm{GL}_n(K) \times D^{\times} \times W_K$ “of index \mathbb{Z} ”. Let \mathcal{H}_c^i be the induction of $H_c^i(\mathcal{M}^n, \overline{\mathbb{Q}}_{\ell})$ from H up to $\mathrm{GL}_n(K) \times D^{\times} \times W_K$.

Theorem 2.2 ([Boy99],[HT01]). *For a π a supercuspidal representation of $\mathrm{GL}_n(K)$ with $\overline{\mathbb{Q}}_{\ell}$ coefficients, there is an isomorphism of ℓ -adic representations of $\mathrm{GL}_n(K) \times W_K$:*

$$\mathrm{Hom}_{D^{\times}}(\mathcal{H}_c^i, \mathrm{JL}(\pi)) \cong \begin{cases} \pi \otimes \sigma^{\#}(\pi), & i = n - 1 \\ 0, & i \neq n - 1 \end{cases}.$$

This form of Thm. 2.2 is the one appearing in [Har02, Thm. 2.3].

The use of an intermediary geometric object to realize the local Langlands and Jacquet-Langlands correspondences may be called the “geometric construction”. This approach has its origins in Deligne’s 1973 letter to Piatetski-Shapiro, and the subsequent paper of Carayol [Car83]. The geometric construction has succeeded in proving Thm. 2.2, but the only known proofs involve realizing F as a completion of a global field and applying techniques of automorphic forms and Shimura varieties.

The geometric construction runs parallel to what might be called the “explicit construction,” which begins with an irreducible Weil parameter σ and attempts to construct its correspondent $\pi(\sigma)$ directly. The idea originates in work of Howe [How77], who constructed a class of “tamely ramified supercuspidals”. The idea is to realize $\pi(\sigma)$ as the induced representation (induction with compact supports) from a finite-dimensional representation ρ of a compact-mod-center subgroup $J \subset \mathrm{GL}_n(K)$. The pairs (J, ρ) are constructed explicitly and *a priori* have nothing to do with geometry. Kutzko used these constructions to complete the proof of Thm. 2.1 for $n = 2$ in [Kut84]. No purely local proof of Thm. 2.1 is known for any particular $n \geq 3$, although the papers [BH05a], [BH05b] give partial results.

2.2 The explicit construction of $\sigma \mapsto \pi(\sigma)$ for certain wild parameters σ of low conductor

Following [BH05a], we run through the explicit construction of $\pi(\sigma)$ for an extremely simple sort of irreducible Weil parameter $\sigma: W_K \rightarrow \mathrm{GL}_n(\mathbb{C})$. We will make every imaginable simplifying assumption on σ short of assuming that σ is tamely ramified. These assumptions are:

1. $\sigma = \mathrm{Ind}_{L/K} \theta$ is induced from a 1-dimensional character θ of $W_L^{\mathrm{ab}} \cong L^\times$, where L/K is an extension of degree n .
2. L/K is unramified.
3. If \mathfrak{p}_L is the maximal ideal of the ring of integers of L , then $\theta(1 + \mathfrak{p}_L^2) = 1$ and $\theta(1 + \mathfrak{p}_L) \neq 1$.
4. The conductor of θ cannot be lowered through twisting by a character of the form $\chi \circ N_{L/L'}$, where $L' \subsetneq L$ is a proper subextension and $N_{L/L'}: L^\times \rightarrow (L')^\times$ is the norm map. (This assumption assures that σ is irreducible.)

Choose an embedding $L \hookrightarrow M_n(K)$, and in doing so view L as a subfield of $M_n(K)$. The induction $\mathrm{Ind}_{L^\times}^{\mathrm{GL}_n(K)} \theta$ is not irreducible; to construct an irreducible representation of $\mathrm{GL}_n(K)$ it is necessary to extend θ to a character of a certain “thickening” J of L^\times . In a sense J will be the largest subgroup of $\mathrm{GL}_n(K)$ containing L^\times to which θ admits a natural extension.

Assume that our embedding $L \hookrightarrow M_n(K)$ is such that $\mathcal{O}_L = M_n(\mathcal{O}_K) \cap L$. Write $M_n(\mathcal{O}_K) = \mathcal{O}_L \oplus C$ where C is the complement to

\mathcal{O}_L under the natural trace pairing on $M_n(\mathcal{O}_K)$. Explicitly, if we identify $M_n(\mathcal{O}_K)$ with the algebra of endomorphisms of the \mathcal{O}_K -module \mathcal{O}_L , then $C = \bigoplus_{i=1}^{n-1} \mathcal{O}_L \sigma^i$, where σ generates $\text{Gal}(L/K)$.

Let $U^1 \subset \text{GL}_n(\mathcal{O}_K)$ be the principal congruence subgroup of matrices congruent to 1 modulo \mathfrak{p}_K , and let $J = L^\times U^1$. Then there exists a unique character $\tilde{\theta}: J \rightarrow \mathbb{C}^\times$ having the properties that $\tilde{\theta}|_{L^\times} = \theta$ and $\theta(1 + \mathfrak{p}_L C) = 1$.

Now let

$$\pi_\theta = \text{Ind}_J^{\text{GL}_n(K)} \tilde{\theta}.$$

Then π_θ is an irreducible supercuspidal representation of $\text{GL}_n(K)$. It is natural to ask whether π_θ is isomorphic to $\pi(\sigma)$. In fact this cannot be true in general. Under the local Langlands correspondence, the determinant $\det \sigma: W_K \rightarrow \mathbb{C}^\times$, considered as a character of K^\times via local class field theory, must agree with the central character of $\pi(\sigma)$. The central character of π_θ is $\theta|_{K^\times}$. But in our situation, a calculation shows that $\det \sigma = (\theta|_{K^\times}) \delta$, where $\delta: K^\times \rightarrow \pm 1$ is the unramified character which takes the value $(-1)^{n-1}$ on a uniformizer of K . (This calculation follows from the fact that if L/K is a finite extension, and V is a virtual representation of W_L of degree 0, then

$$\det(\text{Ind } V) = (\det V) \circ \text{transfer},$$

see [Del73], §1.)

Therefore when n is even, π_θ and $\pi(\sigma)$ cannot agree. The correct formula is

$$\pi(\sigma) = \pi_{\theta \Delta_\theta},$$

where Δ_θ is a character of L^\times , called a “rectifier”. In this particular scenario Δ_θ is an unramified character which takes the value $(-1)^{n-1}$ on a uniformizer of L . If one or more of the simplifying assumptions (2), (3), (4) on σ are lifted, there is still a recipe for constructing a supercuspidal representation π_θ , but the rectifier Δ_θ becomes a much more subtle invariant. Certainly the formula for Δ_θ demands explanation. It is our philosophy that the value of Δ_θ can always be traced to the behavior of a Frobenius eigenvalue on a piece of the cohomology of a variety defined over the residue field of L . See the remark following Thm. 3.2.

2.3 The explicit construction of the Jacquet-Langlands correspondent

Let D/K be a central simple division algebra of dimension n^2 . Let \mathcal{O}_D be the ring of integers of D . In [Bro95] there is a description of the admissible dual of D^\times running parallel to the description of the admissible dual of $\mathrm{GL}_n(K)$ given in [BK93]. The Jacquet-Langlands correspondence (originally constructed by Rogawski [Rog83]) is a bijection JL between irreducible essentially square-integrable representations π of $\mathrm{GL}_n(F)$ and irreducible smooth representations $\pi' = \mathrm{JL}(\pi)$ of D^\times , satisfying

$$\mathrm{tr} \pi(g) = (-1)^{n-1} \mathrm{tr} \pi'(g')$$

for regular elliptic elements $g \in \mathrm{GL}_n(K)$, $g' \in D^\times$ with matching characteristic polynomials.

It is not known in general how to write down the bijection JL in terms of the known parameterizations for the representations of $\mathrm{GL}_n(K)$ and D^\times . Various cases have been worked out; see [Hen93] in the case of prime $n \neq p$, [BH00] in the case $n = p$, [BH05c] in the case that n is a power of p , and [BH11] for a general treatment of the “essentially tame” case.

Now suppose $\sigma = \mathrm{Ind}_{L/K} \theta$ is a Weil parameter for K with the same simplifying assumptions (1)–(4) as in the previous section, and let $\pi = \pi_\theta$. Assume that the invariant of D/K is $1/n$. We now give an explicit construction of $\mathrm{JL}(\pi)$. Along the way we will motivate the construction of the unipotent algebraic group $U^{n,q}$ which plays a central role in our investigation.

Choose an embedding $L \hookrightarrow D$ so that $\mathcal{O}_L = \mathcal{O}_D \cap L$. Then \mathcal{O}_D is generated over \mathcal{O}_K by \mathcal{O}_L and a uniformizer $\varpi \in \mathcal{O}_D$ which satisfies $\varpi^n \in \mathcal{O}_K$ and $\varpi\alpha = \mathrm{Fr}(\alpha)\varpi$, where $\mathrm{Fr} \in \mathrm{Gal}(L/K)$ is the Frobenius element. Once again we have a decomposition $\mathcal{O}_D = \mathcal{O}_L \oplus C'$, where $C' = \bigoplus_{i=1}^{n-1} \mathcal{O}_L \varpi^i$ is the complement of \mathcal{O}_L in \mathcal{O}_D under the (reduced) trace pairing. For $i \geq 1$, let $U_D^i = 1 + \varpi^i \mathcal{O}_D \subset \mathcal{O}_D^\times$ be the usual filtration by principal congruence subgroups. Let θ' be the unique character of U_D^n which satisfies $\theta'|_{U_L^1} = \theta$ and $\theta'(1 + \mathfrak{p}_L C') = 1$. Then θ' factors through a character of U_D^n/U_D^{n+1} .

Now U_D^n/U_D^{n+1} is the center of the group $U := U_D^1/U_D^{n+1}$, which is a unipotent group of order q^{n^2} . Then U is the group consisting of truncated polynomials of the form $1 + a_1\tau + \cdots + a_n\tau^n$, where $a_i \in \mathbf{F}_{q^n}$, $\tau^{n+1} = 0$ and $\tau a = a^q\tau$. Such a polynomial represents the image of the element $1 + \sum_{i=1}^n \tilde{a}_i \varpi^i \in U_B^1$, where $\tilde{a}_i \in \mathcal{O}_L$ is any lift of a_i .

Let $Z \subset U$ be the image of U_D^n , so that $Z \cong \mathbf{F}_{q^n}$ is the center of U . Write ψ for the character of Z induced by the restriction of θ to U_D^n . By assumption (4) on σ , ψ does not factor through any nontrivial trace map $Z \cong \mathbf{F}_{q^n} \rightarrow \mathbf{F}_{q^{n_1}}$. There exists a unique irreducible representation ρ_ψ of U with central character ψ ; for its construction, see Thm. 4.4. We remark in passing that $\dim \rho_\psi = q^{n(n-1)/2}$. The inflation of ρ_ψ to U_D^1 can be extended to a representation η_θ of $J_D = L^\times U_D^1 = L^\times \mathcal{O}_D^\times$ in a unique way such that:

1. For a primitive root of unity $\zeta \in \mathcal{O}_L^\times$ of order $q^n - 1$, we have $\mathrm{tr} \eta_\theta(\zeta) = (-1)^{n-1} \theta(\zeta)$.
2. The central character of η_θ is $\theta|_{K^\times}$.

Then $\pi'_\theta = \mathrm{Ind}_{J_D}^{D^\times} \eta_\theta$ is an irreducible representation of D^\times of dimension $nq^{n(n-1)/2}$.

Proposition 2.3. $\pi'_\theta = \mathrm{JL}(\pi_\theta)$.

Proof. We offer a sketch of a proof which closely follows that given for Théorème 5.1 of [Hen93], which treats the case of n a prime other than the residue characteristic of K . Let $\pi = \pi_\theta$, $\pi' = \mathrm{JL}(\pi)$. Since the Weil parameter $\sigma = \sigma(\pi)$ is induced from a character of L^\times , it must be the case that $\sigma \cong \sigma \otimes \chi$ for every character χ of W_K which factors through $\mathrm{Gal}(L/K)$. By compatibility of the local correspondences with twisting, $\pi' \cong \pi' \otimes (\chi \circ \mathrm{Nm})$ for all unramified characters χ of K^\times which vanish on $N_{L/K}(L^\times)$. (Here Nm is the reduced norm $D^\times \rightarrow K^\times$.) This implies that $\pi' = \pi_{\theta'}$ for some character θ' of L^\times . We must now show that θ' equals one of the K -conjugates of θ .

This requires a comparison of the characters of π and π' . Recall that the field L is embedded in both $M_n(K)$ and D . We say an element $\gamma \in L^\times$ is *very regular* if it belongs to \mathcal{O}_L and if its image modulo \mathfrak{p}_E generates $\mathbf{F}_{q^n}/\mathbf{F}_q$. The images of such elements in $\mathrm{GL}_n(K)$ are automatically regular elliptic, so that the character of π converges on them. The characters of π and π' can be readily calculated on very regular elements of L^\times using Mackey's theorem:

$$\begin{aligned} \mathrm{tr} \pi(\gamma) &= \sum_{g \in \mathrm{Gal}(L/K)} \theta(g(\gamma)) \\ \mathrm{tr} \pi'(\gamma) &= (-1)^{n-1} \sum_{g \in \mathrm{Gal}(L/K)} \theta'(g(\gamma)) \end{aligned}$$

(compare [Hen93], §3.5 Proposition and §4.6 Théorème, noting that the Artin conductor a of θ' is 2.) By definition of the Jacquet-

Langlands correspondence, $\text{tr } \pi(\gamma) = (-1)^{n-1} \text{tr } \pi'(\gamma)$. This is enough to show that θ and θ' are in the same $\text{Gal}(L/K)$ -orbit, cf. the argument in [Hen93], §5.3. \square

3 The main theorem (primitive case)

3.1 Definition of the variety X

Here we construct a variety over a finite field which is the main focus of the paper. We begin by defining a unipotent algebraic group $U^{n,q}$ over \mathbf{F}_q for which the group $U^{n,q}(\mathbf{F}_{q^n})$ is isomorphic to the group $U = U_D^1/U_D^{n+1}$ of the previous section. If A is any commutative \mathbf{F}_q -algebra, the group of A -points $U^{n,q}(A)$ consists of expressions of the form $1 + a_1\tau + \cdots + a_n\tau^n$, where each $a_j \in A$ and τ is a formal symbol. These expressions are multiplied using distributivity and the rules $\tau^{n+1} = 0$ and $\tau \cdot a = a^q \cdot \tau$ for all $a \in A$.

Let $Y_0 \subset U^{n,q}$ be defined by the equation $a_n = 0$, so that $Y_0 \cong \mathbb{A}_{\mathbf{F}_q}^{n-1}$. Write $L_{q^n}: U^{n,q} \rightarrow U^{n,q}$ for the Lang map $g \mapsto \text{Fr}_{q^n}(g) \cdot g^{-1}$, where Fr_{q^n} is the q^n -power Frobenius map. Put $X_0 = L_{q^n}^{-1}(Y_0)$. Let $X = X_0 \otimes_{\mathbf{F}_q} \mathbf{F}_{q^n}$ and $Y = Y_0 \otimes_{\mathbf{F}_q} \mathbf{F}_{q^n}$. Then $U = U^{n,q}(\mathbf{F}_{q^n})$ acts on X by right multiplication and the map $X \rightarrow Y$ induced by L_{q^n} makes X an étale $U^{n,q}(\mathbf{F}_{q^n})$ -torsor over Y . In particular, we obtain an action of $U^{n,q}(\mathbf{F}_{q^n})$ on $H_c^\bullet(X, \overline{\mathbb{Q}}_\ell) := \bigoplus_{i \in \mathbb{Z}} H_c^i(X, \overline{\mathbb{Q}}_\ell)$. In the course of this paper we compute each cohomology group $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ as a representation of U together with the action Fr_{q^n} . We remark that X/\mathbf{F}_{q^n} is the $(n-1)$ -dimensional hypersurface in the variables a_1, \dots, a_n with equation

$$\det \begin{pmatrix} a_1^{q^n} - a_1 & a_2^{q^n} - a_2 & a_3^{q^n} - a_3 & \cdots & a_{n-1}^{q^n} - a_{n-1} & a_n^{q^n} - a_n \\ 1 & a_1^q & a_2^q & \cdots & a_{n-2}^q & a_{n-1}^q \\ 0 & 1 & a_1^{q^2} & \cdots & a_{n-3}^{q^2} & a_{n-2}^{q^2} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_1^{q^{n-1}} \end{pmatrix} = 0.$$

3.2 An open affinoid in the Lubin-Tate tower

Here we summarize the main result of [Wei10]. Recall that we have the decomposition $M_n(\mathcal{O}_K) = \mathcal{O}_L \oplus C$, where C a free \mathcal{O}_L -module of rank $n-1$ which is complementary to $\mathcal{O}_L \subset M_n(\mathcal{O}_K)$ under the trace

pairing. Let $N \subset \mathrm{GL}_n(\mathcal{O}_K)$ be the open subgroup $N = 1 + \mathfrak{p}_L^2 + \mathfrak{p}_L C$. Let $\mathcal{M}_N = \mathcal{M}^n/N$, so that \mathcal{M}_N is intermediate to the covering $\mathcal{M}_2^n \rightarrow \mathcal{M}_1^n$.

Theorem 3.1. *Assume that K has positive characteristic. The rigid space \mathcal{M}_N contains an open affinoid \mathfrak{Z} whose reduction $\overline{\mathfrak{Z}}$ is isomorphic to the variety X of §3.1.*

We remark here that the affinoid \mathfrak{Z} is a neighborhood in \mathcal{M}_N of a “canonical point” corresponding to a formal \mathcal{O}_K -module Σ^{can} with $\mathrm{End} \Sigma^{\mathrm{can}} = \mathcal{O}_L$, whose level structure is determined by a choice of isomorphism $M_n(\mathcal{O}_K) \cong \mathrm{End}_{\mathcal{O}_K} \mathcal{O}_L$ compatible with our chosen embedding $L \hookrightarrow M_n(K)$. (We borrow the term “canonical” from [Gro86].)

3.3 The main theorem, in the case of primitive central character

In order to apply Thm. 3.1 towards a purely local proof of Thm. 2.2 in the positive characteristic case, one needs a description of the ℓ -adic cohomology of X as a module both for the actions of U and for the q^n -power Frobenius. This is done completely in Thm. 4.4. A special case, which is relevant to the discussion in §2.3, is given here.

Let $U = U^{n,q}(\mathbf{F}_{q^n})$, and let $Z \cong \mathbf{F}_{q^n}$ be the center of U .

Call a character $\psi: Z \rightarrow \overline{\mathbb{Q}}_\ell^\times$ *primitive* if it does not factor through any nontrivial trace map $\mathbf{F}_{q^n} \rightarrow \mathbf{F}_{q^{n_1}}$.

Theorem 3.2. *Let ψ be a primitive character of Z . Then the ψ -isotypic component of $H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)$ is an irreducible representation ρ_ψ of U which is supported in the middle degree $n-1$. Furthermore Fr_{q^n} acts on $H_c^{n-1}(X, \overline{\mathbb{Q}}_\ell)$ as the scalar $(-1)^{n-1} q^{n(n-1)/2}$.*

Imprimitive characters of Z do not appear in $H_c^{n-1}(X, \overline{\mathbb{Q}}_\ell)$.

We remark that the factor $q^{n(n-1)/2}$ in the above theorem is related to the Tate twist appearing in Thm. 2.2, and the sign $(-1)^{n-1}$ explains the rectifier Δ_θ appearing in §2.2. These remarks will be explained fully in the following paragraphs.

We also note that there is an action of a second group on X , namely the abelian group $\mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$. An element $\gamma \in \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$ sends $x = 1 + x_1\tau + \cdots + x_n\tau^n$ to

$$“\gamma^{-1}x\gamma” = 1 + \sum_{i=1}^n \gamma^{q^i-1} x_i \tau^i$$

(the quotation marks being necessary because γ does not belong to the group $U^{n,q}$). We therefore have an action of the semidirect product $U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$ on X . The representation ρ_ψ of U on $H_c^{n-1}(X, \overline{\mathbb{Q}_\ell})[\psi]$ extends to a representation of $U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$, which we also call ρ_ψ .

Lemma 3.3. *Let $\gamma \in \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$ be an element such that $\gamma^{q^i-1} \neq 1$ for $i = 1, \dots, n-1$. Then $\mathrm{tr} \rho_\psi(\gamma) = (-1)^{n-1}$.*

Proof. This is a computation with the Lefschetz fixed-point theorem. For $z \in Z$, it is easy to see that $(z, \gamma) \in U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$ only has fixed points on X if $z = 0$, in which case it has q^n fixed points. Writing \mathcal{H}_c^* for the Euler characteristic of X with $\overline{\mathbb{Q}_\ell}$ -coefficients, we find

$$\mathrm{tr}((z, \gamma)|\mathcal{H}_c^{n-1}) = \begin{cases} q^n, & z = 0 \\ 0, & z \neq 0. \end{cases}$$

Since ψ appears in \mathcal{H}_c^* only in degree $n-1$,

$$\begin{aligned} \mathrm{tr} \rho_\psi(\gamma) &= (-1)^{n-1} \mathrm{tr}(\gamma|\mathcal{H}_c^*[\psi]) \\ &= \frac{(-1)^{n-1}}{q^n} \sum_{z \in Z} \psi(z)^{-1} \mathrm{tr}((z, \gamma)|\mathcal{H}_c^{n-1}) \\ &= (-1)^{n-1} \end{aligned}$$

as required. \square

3.4 The relative Weil group

We now explain how the main theorem applies to a purely local approach to a geometric realization of the LLC. The idea is that the affinoid \mathfrak{J} of Thm. 3.1 makes a contribution to the cohomology of \mathcal{M}^n that exactly accounts for the supercuspidal representations π_θ of the sort encountered in §2.2.

Let

$$\mathrm{rec}_L: L^\times \rightarrow W_L^{\mathrm{ab}}$$

be the isomorphism of local class field theory, normalized to send a uniformizer to *geometric* Frobenius. Recall the relative Weil group $W_{L/K}$: this is the quotient of W_K by the closure of $[W_L, W_L]$. The behavior of $W_{L/K}$ is discussed carefully in §1 of [Tat79]. It sits in an exact sequence

$$1 \rightarrow W_L^{\mathrm{ab}} \rightarrow W_{L/K} \rightarrow \mathrm{Gal}(L/K) \rightarrow 1. \quad (3.1)$$

Remark 3.4. Under the isomorphism

$$H^2(\mathrm{Gal}(L/K), W_L^{\mathrm{ab}}) \xrightarrow{\cong} H^2(\mathrm{Gal}(L/K), L^\times)$$

induced by rec_L , the class of the extension in Eq. 3.1 in the group $H^2(\mathrm{Gal}(L/K), W_L^{\mathrm{ab}})$ is mapped to the *inverse* of the fundamental class in $H^2(\mathrm{Gal}(L/K), L^\times)$; see [Tat79, §1.2]. This is a consequence of our choice of the normalization of the reciprocity map rec_L .

Let $\mathcal{N} \subset D^\times$ be the normalizer of L^\times ; then \mathcal{N} is another extension of $\mathrm{Gal}(L/K)$ by L^\times . In fact, this extension represents the same class in $H^2(\mathrm{Gal}(L/K), L^\times)$ as $W_{L/K}$:

Proposition 3.5. *There is an isomorphism $j: W_{L/K} \rightarrow \mathcal{N}$ such that*

$$j(\mathrm{rec}_L(\alpha)) = \alpha$$

for all $\alpha \in L^\times$.

Proof. In view of Remark 3.4 it suffices to check that the cohomology class in $H^2(\mathrm{Gal}(L/K), L^\times)$ of the extension

$$1 \longrightarrow L^\times \longrightarrow \mathcal{N} \longrightarrow \mathrm{Gal}(L/K) \longrightarrow 1 \quad (3.2)$$

is also equal to the inverse of the fundamental class. By definition, the fundamental class is the image of D under the standard isomorphism $\gamma = \gamma_{L/K} : Br(L/K) \xrightarrow{\cong} H^2(\mathrm{Gal}(L/K), L^\times)$, where $Br(L/K)$ is the relative Brauer group of (isomorphism classes of) central division algebras over K that are split by L .

The explicit construction of γ given in [Mil11, p. 132] yields a 2-cocycle of $\mathrm{Gal}(L/K)$ with coefficients in L^\times that represents $\gamma(D)$. It is manifestly obvious that this 2-cocycle is the inverse of the 2-cocycle that represents the class of the extension (3.2). \square

3.5 Manifestation of the correspondences in the cohomology of X

Recall some notations from §2.2. We wrote $M_n(\mathcal{O}_K) = \mathcal{O}_L \oplus C$, where C is the complement to \mathcal{O}_L under the trace pairing. U^1 is the principal congruence subgroup of matrices congruent to 1 modulo \mathfrak{p}_K^i .

We identify $M_n(K)$ with $\mathrm{End}_K L$, so that there is an isomorphism of L -algebras $M_n(K) = \oplus_{\sigma \in \mathrm{Gal}(L/K)} L\sigma$.

Let

$$\varepsilon_G: U^1 \rightarrow Z$$

be the surjective homomorphism which sends $1 + \beta\varpi_K \mapsto \beta \pmod{\mathfrak{p}_L} \in \mathbf{F}_{q^n} \cong Z$, and which sends $1 + \gamma$ to 1 for $\gamma \in \mathfrak{p}_K C$. Let $H = 1 + \mathfrak{p}_L^2 + \mathfrak{p}_L C$ be the kernel of ε_G .

Recall that $\varpi \in D$ was a uniformizer with $\varpi^n = \varpi_K$. Also recall that $U_D^i \subset \mathcal{O}_D^\times$ is the group of elements congruent to 1 modulo ϖ^i . Finally we have $U = U_D^1/U_D^{n+1}$; let

$$\varepsilon_D: U_D^1 \rightarrow U$$

be the quotient map.

Let $\mathcal{J}^0 \subset \mathrm{GL}_n(K) \times D^\times$ be the subgroup $L^\times(U^1 \times U_D^1)$, where $L^\times \subset \mathrm{GL}_n(K) \times D^\times$ is embedded diagonally. There is a unique surjective homomorphism

$$\varepsilon: \mathcal{J}^0 \rightarrow U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$$

with the properties

1. $\varepsilon|_{U^1 \times U_D^1} = \varepsilon_G^{-1} \times \varepsilon_D$
2. $\varepsilon(K^\times) = 1$
3. For $\alpha \in \mathcal{O}_L^\times$, $\varepsilon(\alpha)$ is the image of α under $\mathcal{O}_L^\times \rightarrow \mathbf{F}_{q^n}^\times \rightarrow \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$.

Then the kernel of ε is $K^\times U_L^1(H \times U_D^{n+1})$.

Let $j: W_K \rightarrow W_{L/K} \rightarrow \mathcal{N} \subset D^\times$ be the map of Prop. 3.5. Also let i be the composite map

$$W_K \rightarrow W_{L/K} \rightarrow \mathrm{Gal}(L/K) \hookrightarrow \mathrm{Aut}_K(L) = \mathrm{GL}_n(K).$$

Let $\mathcal{J} \subset \mathrm{GL}_n(K) \times D^\times \times W_{L/K}$ be the subgroup generated by \mathcal{J}^0 and the elements $(i(w), j(w), w)$ for $w \in W_K$. Note that such elements normalize \mathcal{J}^0 , so that $\mathcal{J} \cong \mathcal{J}^0 \rtimes W_K$. We extend the map ε to a map

$$\varepsilon: \mathcal{J} \rightarrow (U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \rtimes \mathbb{Z}$$

by the rules

1. Let $\Phi \in W_K$ be an element with $j(\Phi) = \varpi \in D^\times$. Then $\varepsilon(i(\Phi), j(\Phi), \Phi) = 1 \in \mathbb{Z}$.
2. For $w \in W_L$ with $j(w) = \alpha \in \mathcal{O}_L^\times$, $\varepsilon(1, j(w), w)$ is the image of α in $\mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$.

In order for this to make sense, the generator $1 \in \mathbb{Z}$ acts on $U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$ as the geometric Frobenius on each coordinate.

We have a commutative diagram

$$\begin{array}{ccccccc}
& & 1 & & 1 & & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K^\times U_L^1(H \times U_D^{n+1}) & \longrightarrow & \mathcal{J}^0 & \longrightarrow & U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K^\times U_L^1(H \times U_D^{n+1}) \rtimes I_K & \longrightarrow & \mathcal{J} & \longrightarrow & (U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \rtimes \mathbb{Z} \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & & 1 & & 1
\end{array}$$

In the middle row, I_K is embedded in \mathcal{J} via $w \mapsto (j(w)^{-1}, 1, w)$ ($j(w) \in \mathcal{O}_L^\times$ for $w \in I_K$).

Now let X/\mathbf{F}_{q^n} be the variety from §3.1. As noted in §3.3, X has a natural right action by the group $U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times$. Recall that $X = X_0 \times_{\mathbf{F}_q} \mathbf{F}_{q^n}$ for a variety X_0/\mathbf{F}_q . Let \mathbb{Z} act on X by letting $1 \in \mathbb{Z}$ act as $\text{Fr}_q \times 1$, where $\text{Fr}_q: X_0 \rightarrow X_0$ is the geometric Frobenius map. Then we have an action of $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \rtimes \mathbb{Z}$ on X .

The map ε induces a left action of \mathcal{J} on $H_c^{n-1}(X, \overline{\mathbb{Q}}_\ell)$, which we call \mathcal{H}^{n-1} .

Theorem 3.6. *Let π be an irreducible admissible representation of $\text{GL}_n(K)$, let π' be an irreducible smooth representation of D^\times , and let σ be a continuous irreducible representation of W_K , all with values in $\overline{\mathbb{Q}}_\ell$. The following are equivalent:*

1. $\text{Hom}_{\mathcal{J}}(\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}}, \mathcal{H}^{n-1}) \neq 0$.
2. $\pi' = \text{JL}(\tilde{\pi})$, $\sigma = \sigma^\#(\pi)$, and σ is an n -dimensional Weil parameter satisfying the conditions (1)–(4) of §2.2.

Should either condition hold, the Hom space in statement (1) has dimension 1.

Loosely speaking, this means that the representation of $\text{GL}_n(K) \times D^\times \times W_K$ induced from \mathcal{H}^{n-1} contains exactly those representations

of the form $\pi \otimes \text{JL}(\tilde{\pi}) \otimes \sigma^\#(\pi)$ for those supercuspidal representations $\pi = \pi_\theta$ constructed in §2.2, each with multiplicity one. This state of affairs is entirely consistent with Thm. 2.2.

Proof. Aided by Thm. 3.2, we can give a complete description of \mathcal{H}^{n-1} as a module for the action of $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \rtimes \mathbb{Z}$. Note that $\mathcal{H}^{n-1}[\psi]$ is an irreducible $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times)$ -module, while $1 \in \mathbb{Z}$ carries $\mathcal{H}^{n-1}[\psi]$ onto $\mathcal{H}^{n-1}[\psi^q]$ for each character ψ of Z . Furthermore, $n \in \mathbb{Z}$ acts on \mathcal{H}^{n-1} as the scalar $q^{n(n-1)/2}$. Write δ for the character $n \mapsto (-1)^{n-1} q^{(n-1)/2}$ of $n\mathbb{Z}$. Then for each primitive character ψ , the space $V_\psi = \bigoplus_{i=0}^{n-1} \mathcal{H}^{n-1}[\psi^{q^i}]$ is an irreducible representation of $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \rtimes \mathbb{Z}$ only depending on the $\text{Gal}(\mathbf{F}_{q^n} / \mathbf{F}_q)$ -orbit of ψ . We have that V_ψ is induced from the representation $\rho_\psi \otimes \delta_n$ of $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \times n\mathbb{Z}$.

Let \mathcal{J}^1 be the preimage of $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \times n\mathbb{Z}$ under ε . Then \mathcal{J}^1 is the subgroup of $\text{GL}_n(K) \times D^\times \times W_K$ generated by \mathcal{J}^0 and the elements $(1, \text{rec}_L^{-1} w, w)$, where w runs through W_L . It is isomorphic to $\mathcal{J}^0 \times W_L$.

\mathcal{J}^1 is a normal subgroup of $L^\times U^1 \times L^\times U_D^1 \times W_L$. We claim that if $\lambda \otimes \mu \otimes \chi$ is an irreducible representation of $L^\times U^1 \times L^\times U_D^1 \times W_L$, then the following are equivalent:

- 1' The restriction of $\lambda \otimes \mu \otimes \chi$ to \mathcal{J}^1 is isomorphic to the pullback of $\rho_\psi \otimes \delta_n$ through $\varepsilon: \mathcal{J}^1 \rightarrow (U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \times n\mathbb{Z}$.
- 2' There exists a character θ of L^\times with $\theta(1 + \varpi_K \beta) = \psi(-\beta)$ for all $\beta \in \mathcal{O}_L$, such that $\lambda = \tilde{\theta}$, $\mu = \eta_{\theta^{-1}}$, and $\chi = \theta \Delta_\theta((1-n)/2)$.

(The definitions of $\tilde{\theta}$ and Δ_θ appear in §2.2, and the definition of η_θ appears in §2.3.)

We now explain why the claim implies the theorem. Since V_ψ is induced from $\rho_\psi \otimes \delta_n$, Condition 1 of the theorem is equivalent to the condition that $\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}^1}$ contains the representation of \mathcal{J}^1 obtained by pulling back $\rho_\psi \otimes \delta_n$ through ε , for some primitive character ψ . But then $\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}^1}$ contains some irreducible representation $\lambda \otimes \mu \otimes \chi$ of $L^\times U^1 \times L^\times U_D^1 \times W_L$ which satisfies condition 1'. Applying the claim shows that there exists a character θ such that $\lambda = \tilde{\theta}$, $\mu = \eta_\theta$, and $\chi = \theta \Delta_\theta((1-n)/2)$. But then $\pi \otimes \pi' \otimes \sigma$ is contained in the induction of $\tilde{\theta} \otimes \eta_{\theta^{-1}} \otimes \theta \Delta_\theta((1-n)/2)$ up to $\text{GL}_n(K) \times D^\times \times W_K$, namely $\pi_\theta \otimes \pi'_{\theta^{-1}} \otimes \text{Ind}_{L/K} \theta \Delta_\theta((1-n)/2)$, which is irreducible. Therefore $\pi = \pi_\theta$, $\pi' = \pi'_{\theta^{-1}} = \text{JL}(\tilde{\pi})$, and $\sigma = \sigma^\#(\pi)$ (according to the discussion of §2.2.) As a bonus we find that the multiplicity of $\lambda \otimes \mu \otimes \chi$ in

$\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}^1}$ must be 1, which implies that $\text{Hom}_{\mathcal{J}}(\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}}, \mathcal{H}^{n-1})$ is 1-dimensional.

Conversely, suppose condition 2 of the theorem holds. Then $\pi = \pi_\theta$ for a character θ of L^\times . Then π is induced from $\tilde{\theta}$, π' is induced from $\eta_{\theta^{-1}}$, and σ is induced from $\theta\Delta_\theta((1-n)/2)$. Therefore condition 2' holds for $\tilde{\theta} \otimes \eta_{\theta^{-1}} \otimes \theta\Delta_\theta((1-n)/2)$. Now apply the claim to find that the restriction of $\tilde{\theta} \otimes \eta_{\theta^{-1}} \otimes \theta\Delta_\theta((1-n)/2)$ to \mathcal{J}^1 is isomorphic to the pullback of $\rho_\psi \otimes \delta_n$ through ε . Therefore

$$\text{Hom}_{\mathcal{J}}(\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}}, \varepsilon^* V_\psi) = \text{Hom}_{\mathcal{J}^1}(\pi \otimes \pi' \otimes \sigma|_{\mathcal{J}^1}, \varepsilon^*(\rho_\psi \otimes \delta_n))$$

is nonzero, because $\pi \otimes \pi' \otimes \sigma$ contains a copy of $\varepsilon^* \rho_\psi \otimes \delta_n$, namely $\tilde{\theta} \otimes \eta_{\theta^{-1}} \otimes \theta\Delta_\theta((1-n)/2)$.

We now turn to the proof of the claim. Assume condition 1' obtains. Then $\lambda|_{U^1}$ contains the character which is $1 + \beta\varpi_K \mapsto \psi(-\beta)$ on U_L^1 and trivial on H . Since L^\times normalizes this character of U^1 , $\lambda|_{L^\times}$ must equal a character $\tilde{\theta}$ of $L^\times U^1$, where θ is a character of L^\times for which $\theta(1 + \beta\varpi_K) = \psi(-\beta)$. But then by definition we have $\lambda = \tilde{\theta}$.

Now consider μ . Since $\mu|_{U_D^n}$ contains the character $1 + \beta\varpi^n \mapsto \psi(\beta)$, $\mu|_{U_D^1}$ must contain the pullback of ρ_ψ through ε_D . But since L^\times normalizes this pullback, and μ is irreducible, we must have that $\mu|_{U_D^n}$ actually equals the pullback of ρ_ψ . Therefore there exists a character θ' of L^\times with $\theta'(1 + \beta\varpi_K^n) = \psi(\beta)$ and $\mu = \eta_{\theta'}$. We claim $\theta' = \theta^{-1}$. Since ε is trivial on $K^\times U_L^1 \subset \mathcal{J}^0$, it follows that θ and θ' are inverse to one another on $K^\times U_L^1$. Now let $\gamma \in L^\times$ be a primitive root of unity of order $q^n - 1$. Since $(\theta \otimes \eta_{\theta'})|_{\mathcal{J}^0}$ is isomorphic to the pullback of ρ_ψ through ε , we have (using Lemma 3.3)

$$\begin{aligned} (-1)^{n-1} &= \text{tr } V_\psi(\gamma) \\ &= \text{tr } V_\psi(\varepsilon(\gamma, \gamma, 1)) \\ &= \theta(\gamma) \text{tr } \eta_{\theta'}(\gamma) \\ &= (-1)^{n-1} \theta(\gamma) \theta'(\gamma). \end{aligned}$$

Therefore $\theta' = \theta^{-1}$.

We turn now to χ . For $\alpha \in L^\times$, the element $(\alpha^{-1}, 1, \text{rec } \alpha)$ acts on $\lambda \otimes \mu \otimes \chi$ as the scalar $\theta(\alpha)^{-1} \chi(\alpha)$. But the image of this element under ε is $nd \in \mathbb{Z}$, where d is the valuation of α in L^\times . Since $\lambda \otimes \mu \otimes \chi$ is the pullback of $\rho_\psi \otimes \delta_n$, we must have

$$\theta(\alpha)^{-1} \chi(\alpha) = \delta_n(nd) = ((-1)^{n-1} q^{n(n-1)/2})^d = \Delta_\theta((1-n)/2)(\alpha).$$

Therefore $\chi = \theta\Delta_\theta((1-n)/2)$, and condition 2' holds.

Now assume condition 2' of the claim. The representation $\tau = \tilde{\theta} \otimes \eta_{\theta^{-1}} \otimes \theta\Delta_\theta((1-n)/2)|_{\mathcal{J}^1}$ is visibly trivial on $K^\times U_L^1(H \times U_D^{n+1})$, as well as on the subgroup $I_K \subset \mathcal{J}^1$ embedded via $w \mapsto (j(w)^{-1}, 1, w)$. These groups generated the kernel of $\varepsilon: \mathcal{J}^1 \rightarrow (U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \times n\mathbb{Z}$. Therefore τ is pulled back from an irreducible representation, call it r , of $(U \rtimes \mathbf{F}_{q^n}^\times / \mathbf{F}_q^\times) \times n\mathbb{Z}$. The restriction of r to Z is a sum of copies of ψ . For γ a generator of $\mathbf{F}_{q^n}^\times$, the trace of $r(\gamma)$ equals -1 by a calculation similar to the above. Finally, the element $n \in n\mathbb{Z}$ acts through the scalar $(-1)^{n-1}q^{n(n-1)/2}$. These facts are enough to show that τ is isomorphic to $\rho_\psi \otimes \delta_n$. \square

4 The main theorem (general case)

4.1 General notation

We keep the notation defined in the introduction. Thus p is a prime number, q is a power of p and $n \geq 1$ is an integer. We also choose a prime $\ell \neq p$. We fix an algebraic closure $\overline{\mathbb{Q}_\ell}$ of the field \mathbb{Q}_ℓ of ℓ -adic numbers and an algebraically closed field \mathbf{F} of characteristic p . For any integer $s \geq 1$ we write \mathbf{F}_{p^s} for the unique subfield of \mathbf{F} consisting of p^s elements. Finally, we put $Q = q^n$.

By a *variety* (respectively, an *algebraic group*) we mean a geometrically reduced scheme (respectively, group scheme) of finite type over a field.

Suppose X is a variety over \mathbf{F}_q , and write $\text{pr}: X \rightarrow \text{Spec } \mathbf{F}_q$ for the structure morphism. We put $H_c^i(X, \overline{\mathbb{Q}_\ell}) = R^i \text{pr}_!(\overline{\mathbb{Q}_\ell})$, which is viewed as an ℓ -adic sheaf over $\text{Spec } \mathbf{F}_q$ — in other words, a finite dimensional $\overline{\mathbb{Q}_\ell}$ -vector space with an action of the geometric Frobenius, which we will denote by Fr_q . We remark that the Tate twist $\overline{\mathbb{Q}_\ell}(1)$ corresponds to the vector space $\overline{\mathbb{Q}_\ell}$ on which Fr_q acts as q^{-1} . Similarly, we write $H_c^i(X, \mathcal{F})$ in place of $R^i \text{pr}_!(\mathcal{F})$ for any (constructible) ℓ -adic sheaf \mathcal{F} on X .

By an abuse of notation, we also use the symbol Fr_q in a different context. Namely, if X is a variety over \mathbf{F}_q , we write $\text{Fr}_q: X \rightarrow X$ for the absolute Frobenius morphism (the identity map on the underlying topological space and the map $f \mapsto f^q$ on the structure sheaf). Furthermore, given an integer $m \geq 1$, we will write $\text{Fr}_q: X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^m} \rightarrow$

$X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^m}$ for the morphism obtained from Fr_q by extension of scalars. We remark that with this notation, we have $\text{Fr}_{q^m} = \text{Fr}_q^m$ on $X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^m}$.

In each situation where the symbol Fr_q is used, it will be clear from the context how it must be interpreted, so no confusion should arise.

4.2 Additive characters of \mathbf{F}_Q

Throughout these notes, by a *character* of a group Γ we mean a homomorphism $\Gamma \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Given a character $\psi : \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}}_\ell^\times$ (recall that $Q = q^n$), there exists a unique integer $1 \leq m \leq n$ (which divides n) such that ψ factors through the trace map $\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q^m}} : \mathbf{F}_Q \rightarrow \mathbf{F}_{q^m}$ and does not factor through the trace map $\mathbf{F}_Q \rightarrow \mathbf{F}_{q^k}$ for any $1 \leq k < m$ (cf. §5.4). We call q^m the *conductor* of ψ . Note that since the trace maps for extensions of finite fields are surjective, we can write $\psi = \psi_1 \circ \text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q^m}}$ for a *unique* character $\psi_1 : \mathbf{F}_{q^m} \rightarrow \overline{\mathbb{Q}}_\ell^\times$.

4.3 Preliminaries for the main theorem

We recall that in §3.1 we defined a unipotent group $U^{n,q}$ over \mathbf{F}_q and a variety X over $\mathbf{F}_Q = \mathbf{F}_{q^n}$ together with a right action of $U^{n,q}(\mathbf{F}_Q)$ on X . In Theorem 4.4 we compute each cohomology group $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ as a representation of $U^{n,q}(\mathbf{F}_Q)$ together with the action of the Frobenius Fr_Q on it. First we make some remarks and introduce some additional notation.

Remark 4.1. If $Z \subset U^{n,q}$ consists of expressions of the form $1 + a_n \tau^n$, then Z is the center of $U^{n,q}$ and $Z(\mathbf{F}_Q)$ is the center of $U^{n,q}(\mathbf{F}_Q)$. We have $Z \cong \mathbb{G}_a$, and we often tacitly identify the two groups. In particular, every irreducible representation of $U^{n,q}(\mathbf{F}_Q)$ over $\overline{\mathbb{Q}}_\ell$ has a central character $\mathbf{F}_Q \rightarrow \overline{\mathbb{Q}}_\ell^\times$.

Remark 4.2. As in the introduction, let K be a local nonarchimedean field with residue field \mathbf{F}_q , let L be the unique unramified extension of K of degree n and write $\text{Fr} \in \text{Gal}(L/K)$ for the corresponding Frobenius. Consider the twisted polynomial ring $L\langle\varpi\rangle$, where the relation is $\varpi \cdot x = \text{Fr}(x) \cdot \varpi$ for all $x \in L$. Choose a uniformizer $\pi \in K$ and let D denote the quotient of $L\langle\varpi\rangle$ by the relation $\varpi^n = \pi$. Then D is a central division algebra over K of invariant $1/n$, and ϖ is a uniformizer in D . Recall also that we denote by $\mathcal{O}_D \subset D$ the ring of integers and by $\{U_D^i = 1 + \varpi^i \mathcal{O}_D\}_{i \geq 1}$ the usual filtration of \mathcal{O}_D^\times by

principal congruence subgroups. As mentioned in §3.1, the quotient group U_D^1/U_D^{n+1} can be naturally identified with $U^{n,q}(\mathbf{F}_Q)$.

The reduced norm homomorphism $D^\times \longrightarrow K^\times$ induces a map

$$U_D^1/U_D^{n+1} = (1 + \varpi \mathcal{O}_D)/(1 + \varpi^{n+1} \mathcal{O}_D) \longrightarrow (1 + \pi \mathcal{O}_K)/(1 + \pi^2 \mathcal{O}_K).$$

The right hand side is naturally isomorphic to the additive group of \mathbf{F}_q , so we obtain a group homomorphism $\text{Nm}^{n,q} : U^{n,q}(\mathbf{F}_Q) \longrightarrow \mathbf{F}_q$. It is independent of the choice of K by Lemma 6.1. By a slight abuse of terminology, we will also refer to this homomorphism as the reduced norm map.

This map plays the following role in the study of representations of $U^{n,q}(\mathbf{F}_Q)$. The restriction of $\text{Nm}^{n,q}$ to $Z(\mathbf{F}_Q) = \mathbf{F}_Q$ is equal to the trace map $\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}$. In particular, given a character $\psi : Z(\mathbf{F}_Q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ with conductor q , we obtain a preferred extension of ψ to a character of $U^{n,q}(\mathbf{F}_Q)$. Namely, if $\psi = \psi_1 \circ \text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}$, where $\psi_1 : \mathbf{F}_q \longrightarrow \overline{\mathbb{Q}}_\ell^\times$, then $\psi_1 \circ \text{Nm}^{n,q} : U^{n,q}(\mathbf{F}_Q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ extends ψ .

Remark 4.3. Suppose that $n = m \cdot n_1$, where $m, n_1 \in \mathbb{N}$, and put $q_1 = q^m$, so that $q_1^{n_1} = Q = q^n$. We can consider the unipotent group U^{n_1, q_1} over \mathbf{F}_{q_1} . To avoid confusion, let us temporarily denote its elements by $1 + b_1 \tau_1 + \cdots + b_{n_1} \tau_1^{n_1}$. We can naturally embed U^{n_1, q_1} as a subgroup of $U^{n,q} \otimes_{\mathbf{F}_q} \mathbf{F}_{q_1}$ via the map

$$1 + b_1 \tau_1 + b_2 \tau_1^2 + \cdots + b_{n_1} \tau_1^{n_1} \longmapsto 1 + b_1 \tau^m + b_2 \tau^{2m} + \cdots + b_{n_1} \tau^n.$$

From now on we identify U^{n_1, q_1} with its image under this embedding. In particular, we view $U^{n_1, q_1}(\mathbf{F}_Q)$ as the subgroup of $U^{n,q}(\mathbf{F}_Q)$ consisting of all elements of the form $1 + \sum_{m|j} a_j \tau^j$, where each $a_j \in \mathbf{F}_Q$.

4.4 Statement of the main theorem

Theorem 4.4. *Fix an arbitrary character $\psi : \mathbf{F}_Q \longrightarrow \overline{\mathbb{Q}}_\ell^\times$.*

- (a) *There is a unique (up to isomorphism) irreducible representation ρ_ψ of $U^{n,q}(\mathbf{F}_Q)$ with central character ψ that occurs in*

$$H_c^\bullet(X, \overline{\mathbb{Q}}_\ell) := \bigoplus_{i \in \mathbb{Z}} H_c^i(X, \overline{\mathbb{Q}}_\ell).$$

Moreover, the multiplicity of ρ_ψ in $H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)$ as a representation of $U^{n,q}(\mathbf{F}_Q)$ is equal to 1.

- (b) Let ψ have conductor q^m , so that $n = mn_1$ for some $n_1 \in \mathbb{N}$. Then ρ_ψ occurs in $H_c^{n+n_1-2}(X, \overline{\mathbb{Q}_\ell})$, and Fr_Q acts on it via the scalar $(-1)^{n-n_1} \cdot q^{n(n+n_1-2)/2}$.
- (c) The representation ρ_ψ can be constructed as follows. Write $\psi = \psi_1 \circ \text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q_1}}$ for a unique character $\psi_1 : \mathbf{F}_{q_1} \rightarrow \overline{\mathbb{Q}_\ell}^\times$, where $q_1 = q^m$ as in Remark 4.3. Put

$$H_m = \left\{ 1 + \sum_{\substack{j \leq n/2 \\ m|j}} a_j \tau^j + \sum_{n/2 < j \leq n} a_j \tau^j \right\} \subset U^{n,q},$$

a connected subgroup. The projection $\nu_m : H_m \rightarrow U^{n_1, q_1}$ obtained by discarding all summands $a_j \tau^j$ with $m \nmid j$ (cf. Remark 4.3) is a group homomorphism, and $\tilde{\psi} := \psi_1 \circ \text{Nm}^{n_1, q_1} \circ \nu_m$ is a character of $H_m(\mathbf{F}_Q)$ that extends $\psi : Z(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}_\ell}^\times$ (see Remark 4.2). With this notation:

- if m is odd or n_1 is even, then $\rho_\psi \cong \text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$;
- if m is even and n_1 is odd, then $\text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$ is isomorphic to a direct sum of $q^{n/2}$ copies of ρ_ψ . Moreover, in this case, if $\Gamma_m \subset U^{n,q}(\mathbf{F}_Q)$ is the subgroup consisting of all elements of the form $h + a_{n/2} \tau^{n/2}$, where $h \in H_m(\mathbf{F}_Q)$ and $a_{n/2} \in \mathbf{F}_{q^{n/2}}$, then $\tilde{\psi}$ can be extended to a character of Γ_m , and if $\chi : \Gamma_m \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is any such extension, then $\rho_\psi \cong \text{Ind}_{\Gamma_m}^{U^{n,q}(\mathbf{F}_Q)}(\chi)$.

The proof of this result is given in Section 7.

Examples 4.5. In order to clarify the construction of Theorem 4.4(c), let us consider two special cases. Assume first that $m = 1$ and $n_1 = n$. This means that we can write $\psi = \psi_1 \circ \text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}$ for a (unique) character $\psi_1 : \mathbf{F}_q \rightarrow \overline{\mathbb{Q}_\ell}^\times$. In this case $H_1 = U^{n_1, q_1} = U^{n,q}$ and $\rho_\psi = \tilde{\psi} = \psi_1 \circ \text{Nm}^{n,q}$ is 1-dimensional.

Next consider the case where $m = n$ and $n_1 = 1$. Thus $\psi : Z(\mathbf{F}_Q) = \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is a character that does not factor through the trace map $\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q^k}}$ for any $1 \leq k < n$. In this case $U^{n_1, q_1} = Z$ and $H_n \subset U^{n,q}$ consists of all elements of the form $h = 1 + \sum_{j > n/2} a_j \tau^j$. The character $\tilde{\psi} : H_n(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is given by $\tilde{\psi}(h) = \psi(a_n)$.

If n is odd, then $\rho_\psi = \text{Ind}_{H_n(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$. If n is even, then we need to consider the subgroup $\Gamma_n \subset U^{n,q}(\mathbf{F}_Q)$ consisting of elements of the form $\gamma = 1 + \sum_{j \geq n/2} a_j \tau^j$, where $a_{n/2} \in \mathbf{F}_{q^{n/2}}$ and $a_j \in \mathbf{F}_Q$ for all $j > n/2$. Then $\tilde{\psi}$ extends to a character $\chi : \Gamma_n \rightarrow \overline{\mathbb{Q}_\ell}^\times$ (if q is odd, we can take $\chi(\gamma) = \psi(a_n - \frac{a_{n/2}^2}{2})$; if q is even, an extension χ also exists, but has to be defined differently), and we have $\rho_\psi \cong \text{Ind}_{\Gamma_n}^{U^{n,q}(\mathbf{F}_Q)}(\chi)$.

If $m = n$, then whether n is even or odd, one can show that ρ_ψ is the unique (up to isomorphism) irreducible representation of $U^{n,q}(\mathbf{F}_Q)$ with central character ψ .

Corollary 4.6 (Cohomology of X). *We have $H_c^i(X, \overline{\mathbb{Q}_\ell}) = 0$ unless $i = n + n_1 - 2$ for some $1 \leq n_1 \leq n$ such that $n_1 \mid n$. In the latter case Fr_Q acts on $H_c^{n+n_1-2}(X, \overline{\mathbb{Q}_\ell})$ via the scalar $(-1)^{n-n_1} \cdot q^{n(n+n_1-2)/2}$, and we have*

$$\dim H_c^{n+n_1-2}(X, \overline{\mathbb{Q}_\ell}) = q^{n(n-n_1)/2} \cdot \sum_{d \mid m} \mu(d) q^{m/d}, \quad (4.1)$$

where $n = n_1 \cdot m$ and μ is the Möbius function.

Proof. All the assertions save (4.1) follow from parts (a) and (b) of Theorem 4.4. To prove (4.1) we first observe that the number of characters $\psi : \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}_\ell}^\times$ with conductor q^m is the same as the number of elements in \mathbf{F}_{q^m} that do not belong to \mathbf{F}_{q^k} for any $1 \leq k < m$ (cf. §5.4). That number is $\sum_{d \mid m} \mu(d) q^{m/d}$. It remains to calculate $\dim \rho_\psi$ for any such ψ . To this end, we note that the index of $U^{n_1, q_1}(\mathbf{F}_Q)$ in $U^{n,q}(\mathbf{F}_Q)$ is given by $(U^{n,q}(\mathbf{F}_Q) : U^{n_1, q_1}(\mathbf{F}_Q)) = Q^{n-n_1} = q^{n(n-n_1)}$ and ρ_ψ is induced from a character of a subgroup of $U^{n,q}(\mathbf{F}_Q)$ of index $(U^{n,q}(\mathbf{F}_Q) : U^{n_1, q_1}(\mathbf{F}_Q))^{1/2}$. Thus $\dim \rho_\psi = q^{n(n-n_1)/2}$, and in view of the multiplicity 1 assertion of Theorem 4.4(a), formula (4.1) follows. \square

Corollary 4.7 (Maximality). *In the setup above X is a maximal variety over \mathbf{F}_Q in the sense defined in the introduction.*

Proof. By Cor. 4.6, for any $i \in \mathbb{Z}$ such that $H_c^i(X, \overline{\mathbb{Q}_\ell}) \neq 0$, the Frobenius Fr_Q acts on $H_c^i(X, \overline{\mathbb{Q}_\ell})$ via the scalar $(-1)^i \cdot Q^{i/2}$, whence the claim. \square

Remark 4.8. The restriction of the Lang map L_Q to X realizes X as a finite étale cover of the subvariety $\{1 + a_1\tau + \cdots + a_n\tau^n \mid a_n = 0\} \subset U^{n,q} \otimes_{\mathbf{F}_q} \mathbf{F}_Q$, which is isomorphic to \mathbb{A}^{n-1} . Hence all connected components of $X \otimes_{\mathbf{F}_Q} \overline{\mathbf{F}}_Q$ are irreducible and smooth of dimension $n - 1$. By Corollary 4.6, the top compactly supported cohomology $H_c^{2n-2}(X, \overline{\mathbb{Q}}_\ell)$ has dimension q and Fr_Q acts on it via the scalar Q^{n-1} . This implies that X has q connected components, which are geometrically irreducible and smooth of dimension $n - 1$.

Let us give an explicit description of these components. In §6.1 below we introduce a morphism $N^{n,q} : U^{n,q} \longrightarrow \mathbb{G}_a$, which extends the reduced norm map $\text{Nm}^{n,q}$ in the sense that $N^{n,q} : U^{n,q}(\mathbf{F}_Q) \longrightarrow \mathbf{F}_Q$ has image in \mathbf{F}_q and is equal to $\text{Nm}^{n,q}$. By Proposition 6.2, X can be described as the subvariety of $U^{n,q} \otimes_{\mathbf{F}_q} \mathbf{F}_Q$ defined by the equation $N^{n,q}(g)^q = N^{n,q}(g)$. Hence the connected components of X are precisely the subvarieties given by $N^{n,q}(g) = c$ as c ranges over the points of $\mathbf{F}_q \subset \mathbb{G}_a$.

5 Some preliminaries

5.1 Representations of finite groups

Throughout this text by ‘representation’ we mean “representation of a finite group on a finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector space.”

Lemma 5.1. *Let Γ be a finite group, $N \subset \Gamma$ a normal subgroup and $\chi : N \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ a character. Write $\Gamma^\chi \subset \Gamma$ for the stabilizer of χ with respect to the conjugation action of Γ . If ρ is any irreducible representation of Γ^χ whose restriction to N is isomorphic to a direct sum of copies of χ , then $\text{Ind}_{\Gamma^\chi}^\Gamma \rho$ is irreducible.*

Proof. We verify the hypothesis of Mackey’s irreducibility criterion. Choose $\gamma \in \Gamma$ with $\gamma \notin \Gamma^\chi$. Write ρ^γ for the representation of $\gamma\Gamma^\chi\gamma^{-1}$ given by $g \mapsto \rho(\gamma^{-1}g\gamma)$. Then $N \subset \Gamma^\chi \cap \gamma\Gamma^\chi\gamma^{-1}$, and since γ does not normalize χ , the assumption of the lemma implies that the restrictions of ρ and ρ^γ to N have no irreducible components in common. *A fortiori*, the restrictions of ρ and ρ^γ to $\Gamma^\chi \cap \gamma\Gamma^\chi\gamma^{-1}$ have no irreducible components in common. \square

For the next lemma, see [BD06, Prop. B.4] and its proof.

Lemma 5.2. *Let H be a finite group, $N \subset H$ a normal subgroup and $\chi : N \rightarrow \overline{\mathbb{Q}}_\ell^\times$ a character that is invariant under H -conjugation. Assume moreover that N contains the commutator subgroup (H, H) of H .*

- (a) *The map $H \times H \rightarrow \overline{\mathbb{Q}}_\ell^\times$ given by $(h_1, h_2) \mapsto \chi(h_1 h_2 h_1^{-1} h_2^{-1})$ descends to a bimultiplicative map $B_\chi : (H/N) \times (H/N) \rightarrow \overline{\mathbb{Q}}_\ell^\times$.*
- (b) *Let $K = \{x \in H/N \mid B_\chi(x, y) = 1 \ \forall y \in H/N\}$ denote the kernel of B_χ and write $K' \subset H$ for the preimage of K in H . Then χ extends to a character of K' , and given any such extension $\chi' : K' \rightarrow \overline{\mathbb{Q}}_\ell^\times$, the induced representation $\text{Ind}_{K'}^H \chi'$ is a direct sum of copies of an irreducible representation $\rho_{\chi'}$ of H .*
- (c) *Let $L \subset H/N$ be maximal among all subgroups of H/N with the property that $B_\chi|_{L \times L} \equiv 1$, write $L' \subset H$ for the preimage of L in H (so that $K' \subset L'$), and let $\chi' : K' \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be as in part (b). Then χ' extends to a character of L' , and given any such extension $\tilde{\chi}' : L' \rightarrow \overline{\mathbb{Q}}_\ell^\times$, we have $\rho_{\chi'} \cong \text{Ind}_{L'}^H \tilde{\chi}'$.*

Remark 5.3. In the setting of the lemma, the order $\#(H/K')$ of H/K' is necessarily a square, and any $L' \subset H$ as in part (c) has the property that $\#(L'/K') = \#(H/K')^{1/2}$.

5.2 ℓ -adic cohomology

Our conventions regarding ℓ -adic sheaves, cohomology and the sheaves-to-functions correspondence are the same as those of [Del77]. We call a “ $\overline{\mathbb{Q}}_\ell$ -local system” what is called a “ $\overline{\mathbb{Q}}_\ell$ -faisceau lisse” in *op. cit.* If X is a variety over \mathbf{F}_q and \mathcal{F} is a constructible $\overline{\mathbb{Q}}_\ell$ -sheaf on X , the function corresponding to \mathcal{F} will be denoted by $t_{\mathcal{F}} : X(\mathbf{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell$.

We write $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ for the bounded derived category of complexes of constructible $\overline{\mathbb{Q}}_\ell$ -sheaves on X . In fact, throughout this text we will only encounter complexes concentrated in a single cohomological degree. As usual, the square brackets $[\]$ denote cohomological shifts and the parentheses $(\)$ denote Tate twists.

If $f : X \rightarrow Y$ is a morphism of varieties over \mathbf{F}_q , we have the pullback functor $f^* : D_c^b(Y, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and the functor of pushforward with compact supports $Rf_! : D_c^b(X, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(Y, \overline{\mathbb{Q}}_\ell)$. Both are compatible with the sheaves-to-functions correspondence in the natural sense.

Two results are used in many places in this article: the proper base change theorem and the projection formula. We recall their statements.

Theorem 5.4 (Proper base change). *If*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram of varieties over \mathbf{F}_q , there is a canonical isomorphism of functors

$$g^* \circ (Rf_!) \xrightarrow{\cong} Rf'_! \circ g'^* : D_c^b(X, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^b(Y', \overline{\mathbb{Q}}_\ell).$$

Theorem 5.5 (Projection formula). *If $f : X \longrightarrow Y$ is a morphism of varieties over \mathbf{F}_q , then for $M \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and $N \in D_c^b(Y, \overline{\mathbb{Q}}_\ell)$, we have a natural isomorphism*

$$Rf_!((f^*N) \otimes M) \cong N \otimes Rf_!(M) \quad \text{in } D_c^b(Y, \overline{\mathbb{Q}}_\ell).$$

These theorems are standard (see, e.g., SGA4, SGA4 $\frac{1}{2}$, SGA5).

5.3 Multiplicative local systems

Let G be an algebraic group over \mathbf{F}_q . A rank 1 $\overline{\mathbb{Q}}_\ell$ -local system \mathcal{L} on G is said to be *multiplicative* if $\mu^*(\mathcal{L}) \cong \text{pr}_1^*(\mathcal{L}) \otimes \text{pr}_2^*(\mathcal{L})$, where $\mu : G \times G \longrightarrow G$ is the group operation and $\text{pr}_1, \text{pr}_2 : G \times G \longrightarrow G$ are the two projections. In this case the corresponding trace-of-Frobenius function takes values in $\overline{\mathbb{Q}}_\ell^\times$ and is a character $t_{\mathcal{L}} : G(\mathbf{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$.

Lemma 5.6. *If G is a connected commutative algebraic group over \mathbf{F}_q , the map $[\mathcal{L}] \mapsto t_{\mathcal{L}}$ is a bijection between the set of isomorphism classes of multiplicative local systems on G and $\text{Hom}(G(\mathbf{F}_q), \overline{\mathbb{Q}}_\ell^\times)$.*

This follows from Lemmas 1.10 and 1.11 in [BD06].

Lemma 5.7. *Let G be a connected algebraic group over \mathbf{F}_q . If \mathcal{L} is any nontrivial multiplicative local system on G , then $H_c^i(G, \mathcal{L}) = 0$ for all $i \in \mathbb{Z}$.*

For the proof, see [Boy10, Lem. 9.4].

5.4 Multiplicative local systems on \mathbb{G}_a

Let us fix, once and for all, a nontrivial character $\psi_0 : \mathbf{F}_p \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. For each $k \in \mathbb{N}$, the map $\mathbf{F}_{p^k} \longrightarrow \text{Hom}(\mathbf{F}_{p^k}, \overline{\mathbb{Q}}_\ell^\times)$ taking $a \in \mathbf{F}_{p^k}$ to the homomorphism $x \mapsto \psi_0(\text{Tr}_{\mathbf{F}_{p^k}/\mathbf{F}_p}(ax))$ is a group isomorphism, which is compatible with the action of the Frobenius Fr_p on both sides because $\text{Tr}_{\mathbf{F}_{p^k}/\mathbf{F}_p}(a^p x) = \text{Tr}_{\mathbf{F}_{p^k}/\mathbf{F}_p}(ax^{1/p})$ for all $a, x \in \mathbf{F}_{p^k}$.

Next, if $k, r \in \mathbb{N}$ are such that $k \mid r$, then the diagram

$$\begin{array}{ccc} \mathbf{F}_{p^r} & \longrightarrow & \text{Hom}(\mathbf{F}_{p^r}, \overline{\mathbb{Q}}_\ell^\times) \\ \text{inclusion} \uparrow & & \uparrow \text{composition with } \text{Tr}_{\mathbf{F}_{p^r}/\mathbf{F}_{p^k}} \\ \mathbf{F}_{p^k} & \longrightarrow & \text{Hom}(\mathbf{F}_{p^k}, \overline{\mathbb{Q}}_\ell^\times) \end{array}$$

commutes, where the horizontal arrows are the identifications described in the previous paragraph. This justifies the definition of the conductor of a character $\mathbf{F}_Q \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ given in §4.2: indeed, given $a \in \mathbf{F}_Q$, there exists a unique integer $1 \leq m \leq n$ such that $a \in \mathbf{F}_{q^m}$ and $a \notin \mathbf{F}_{q^k}$ for any $1 \leq k < n$; moreover, $m \mid n$.

For this subsection, let \mathbb{G}_a denote the additive group over $\text{Spec } \mathbf{F}_p$. By Lemma 5.6, there exists a unique (up to isomorphism) multiplicative local system \mathcal{L}_{ψ_0} on \mathbb{G}_a whose trace-of-Frobenius function is equal to $\psi_0 : \mathbf{F}_p \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. We can also consider the base change of \mathcal{L}_{ψ_0} to the algebraic closure $\overline{\mathbf{F}}_p$ of \mathbf{F}_p in \mathbf{F} . For every $x \in \overline{\mathbf{F}}_p$, let \mathcal{L}_x denote the pullback of \mathcal{L}_{ψ_0} by the morphism $y \mapsto xy$. Then \mathcal{L}_x is a local system on $\mathbb{G}_a \otimes \overline{\mathbf{F}}_p$, and if $x \in \mathbf{F}_{p^k}$ for some integer $k \geq 1$, then \mathcal{L}_x is also defined over \mathbf{F}_{p^k} .

Lemma 5.8. *For each integer $k \geq 1$, the map $x \mapsto \mathcal{L}_x$ is an isomorphism between \mathbf{F}_{p^k} and the group² of (isomorphism classes of) multiplicative local systems on $\mathbb{G}_a \otimes \mathbf{F}_{p^k}$. The character $\mathbf{F}_{p^k} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ corresponding to \mathcal{L}_x is given by $y \mapsto \psi_0(\text{Tr}_{\mathbf{F}_{p^k}/\mathbf{F}_p}(xy))$.*

This result follows from the previous remarks and Lemma 5.6.

Our next goal is to compute the action of the ring of algebraic group endomorphisms $\text{End}_{\overline{\mathbf{F}}_p}(\mathbb{G}_a \otimes \overline{\mathbf{F}}_p)$ on the group of isomorphism classes of multiplicative local systems on $\mathbb{G}_a \otimes \overline{\mathbf{F}}_p$. Let us make this

²The group operation is given by the tensor product of local systems.

problem more precise. The last lemma yields a natural identification of this group with the additive group of $\overline{\mathbf{F}}_p$. For each $f \in \text{End}_{\overline{\mathbf{F}}_p}(\mathbb{G}_a \otimes \overline{\mathbf{F}}_p)$ and each $x \in \overline{\mathbf{F}}_p$, the pullback $f^*(\mathcal{L}_x)$ is another multiplicative local system on $\mathbb{G}_a \otimes \overline{\mathbf{F}}_p$, whence $f^*(\mathcal{L}_x) \cong \mathcal{L}_{f^*(x)}$ for a unique element $x \in \overline{\mathbf{F}}_p$, and $f^* : \overline{\mathbf{F}}_p \rightarrow \overline{\mathbf{F}}_p$ is an additive homomorphism. We would like to calculate the map $f \mapsto f^*$ explicitly.

To this end, we observe that every element of $\text{End}_{\overline{\mathbf{F}}_p}(\mathbb{G}_a \otimes \overline{\mathbf{F}}_p)$ can be written uniquely as $a_0 + a_1\tau_p + \cdots + a_d\tau_p^d$, where $a_j \in \overline{\mathbf{F}}_p$ and $\tau_p : \mathbb{G}_a \rightarrow \mathbb{G}_a$ is the endomorphism $x \mapsto x^p$. Furthermore, the map $f \mapsto f^*$ is an antihomomorphism, that is, it is additive and $(f \circ g)^* = g^* \circ f^*$. So it suffices to calculate f^* when f is a scalar in $\overline{\mathbf{F}}_p$, and when $f = \tau_p$. The answer is provided by the following lemma.

Lemma 5.9. *If $f \in \text{End}_{\overline{\mathbf{F}}_p}(\mathbb{G}_a \otimes \overline{\mathbf{F}}_p)$ is the endomorphism of multiplication by some element of $\overline{\mathbf{F}}_p$, then $f^* = f$. Moreover, $\tau_p^* = \tau_p^{-1}$ is the map $x \mapsto x^{1/p}$.*

Proof. If $z \in \overline{\mathbf{F}}_p$ is such that $f(x) = xz$ for all x , then $f^*(\mathcal{L}_x)$ is the pullback of \mathcal{L}_{ψ_0} via the map $y \mapsto zxy$, which implies that $f^*(x) = zx$, proving the first assertion of the lemma. On the other hand, if $f(x) = x^p$, then $f^*(\mathcal{L}_x)$ is the pullback of \mathcal{L}_{ψ_0} by the map $y \mapsto x \cdot y^p = (x^{1/p} \cdot y)^p$. Since \mathcal{L}_{ψ_0} is defined over \mathbf{F}_p , it is invariant under pullback via $y \mapsto y^p$, which means that $f^*(\mathcal{L}_x)$ is isomorphic to the pullback of \mathcal{L}_{ψ_0} via $y \mapsto x^{1/p}y$. So $f^*(\mathcal{L}_x) \cong \mathcal{L}_{x^{1/p}}$, which proves the lemma. \square

Corollary 5.10. *We have $(a_0 + a_1\tau_p + \cdots + a_d\tau_p^d)^* = a_0 + a_1^{1/p}\tau_p^{-1} + \cdots + a_d^{1/p^d}\tau_p^{-d}$.*

Remark 5.11. Suppose f is an endomorphism of $\mathbb{G}_a \otimes \mathbf{F}_{p^k}$ over \mathbf{F}_{p^k} . Then we can consider the endomorphism of $\text{Hom}(\mathbf{F}_{p^k}, \overline{\mathbb{Q}}_\ell^\times)$ given by $\psi \mapsto \psi \circ f$, which induces an additive map $f^* : \mathbf{F}_{p^k} \rightarrow \mathbf{F}_{p^k}$ using the identification $\mathbf{F}_{p^k} \xrightarrow{\cong} \text{Hom}(\mathbf{F}_{p^k}, \overline{\mathbb{Q}}_\ell^\times)$ mentioned earlier. It is clear that in this setting we can also calculate f^* using the formula of Corollary 5.10.

5.5 The Lang isogeny

Let G be an algebraic group over \mathbf{F}_q . The Lang isogeny for G is the map $L_q : G \rightarrow G$ given by $L_q(g) = \text{Fr}_q(g) \cdot g^{-1}$. We will only consider this map for connected G , in which case L_q is a surjective

finite étale map. Moreover, L_q then identifies G with the quotient of G by the right multiplication action of the finite discrete group $G(\mathbf{F}_q)$. We assume from now on that G is connected and view G as a right $G(\mathbf{F}_q)$ -torsor over itself by means of L_q , which we call the “Lang torsor.”

If ρ is a representation of $G(\mathbf{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, we denote by \mathcal{E}_ρ the $\overline{\mathbb{Q}}_\ell$ -local system associated to the Lang torsor by means of ρ . For the definition of \mathcal{E}_ρ , see [Del77, §§1.2 and 1.22 in *Sommes Trig.*]; note that \mathcal{E}_ρ is denoted by $\mathcal{F}(\rho)$ in *loc. cit.*

Proposition 5.12. *Let $\widehat{G(\mathbf{F}_q)}$ be a set of representatives of the isomorphism classes of all irreducible representations of $G(\mathbf{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. Then*

$$L_{q!}(\overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\rho \in \widehat{G(\mathbf{F}_q)}} \rho \otimes \mathcal{E}_\rho \quad (5.1)$$

as local systems with an action of $G(\mathbf{F}_q)$, where $\overline{\mathbb{Q}}_\ell$ is the constant sheaf on G and the action of $G(\mathbf{F}_q)$ on the pushforward $L_{q!}(\overline{\mathbb{Q}}_\ell) = L_{q*}(\overline{\mathbb{Q}}_\ell)$ comes from the right multiplication action of $G(\mathbf{F}_q)$ on G .

We remark that in formula (5.1), the action of $G(\mathbf{F}_q)$ on each of the summands $\rho \otimes \mathcal{E}_\rho$ comes only from the action of $G(\mathbf{F}_q)$ on ρ .

Corollary 5.13. *Let $Y \subset G$ be an \mathbf{F}_q -subvariety and put $X = L_q^{-1}(Y)$. Then*

$$\mathrm{Hom}_{G(\mathbf{F}_q)}(\rho, H_c^i(X, \overline{\mathbb{Q}}_\ell)) \cong H_c^i(Y, \mathcal{E}_\rho|_Y) \quad (5.2)$$

as vector spaces with an action of Fr_q , for any $i \in \mathbb{Z}$ and any representation ρ of $G(\mathbf{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, where the action of $G(\mathbf{F}_q)$ on $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ comes from the right multiplication action of $G(\mathbf{F}_q)$ on X .

Proof. We have $H_c^i(X, \overline{\mathbb{Q}}_\ell) \cong H_c^i(Y, L_{q!}(\overline{\mathbb{Q}}_\ell)|_Y)$ by the proper base change theorem. Both sides of (5.2) are additive with respect to ρ , so it suffices to prove it when ρ is irreducible. In that case (5.2) follows from (5.1). \square

Lemma 5.14. *If G is a connected commutative algebraic group over \mathbf{F}_q , then for every character $\chi : G(\mathbf{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, the corresponding local system \mathcal{E}_χ is multiplicative and its trace-of-Frobenius function $t_{\mathcal{E}_\chi}$ is equal to χ .*

We also need to be able to calculate the trace-of-Frobenius function $t_{\mathcal{E}_\rho}$ more generally. The following result is proved in [Del77, §1.23 in *Sommes Trig.*].

Proposition 5.15. *Given $\gamma \in G(\mathbf{F}_q)$, choose any $g \in G(\overline{\mathbf{F}}_q)$ with $\gamma = L_q(g)$. Then $g^{-1} \cdot \text{Fr}_q(g) \in G(\mathbf{F}_q)$ and $t_{\mathcal{E}_\rho}(\gamma) = \text{tr}(\rho(g^{-1} \cdot \text{Fr}_q(g)))$.*

Our final goal is to relate the construction $\rho \mapsto \mathcal{E}_\rho$ with the operation of induction of group representations.

Proposition 5.16. *Let G be a connected algebraic group over \mathbf{F}_q , and let $H \subset G$ be a connected algebraic subgroup. Let η be a representation of $H(\mathbf{F}_q)$ over $\overline{\mathbb{Q}}_\ell$, put $\rho = \text{Ind}_{H(\mathbf{F}_q)}^{G(\mathbf{F}_q)} \eta$, and write \mathcal{E}_η (respectively, \mathcal{E}_ρ) for the $\overline{\mathbb{Q}}_\ell$ -local system on H (respectively, on G) coming from the Lang isogeny for H (respectively, for G) via η (respectively, via ρ), as above.*

Suppose that $s : G/H \rightarrow G$ is a section³ of the quotient map $G \rightarrow G/H$. Write

$$\text{pr}_2 : (G/H) \times H \rightarrow H$$

for the second projection, and define

$$F : (G/H) \times H \rightarrow G$$

by

$$F(x, h) = \text{Fr}_q(s(x)) \cdot h \cdot s(x)^{-1}.$$

Then

$$\mathcal{E}_\rho \cong F_!(\text{pr}_2^* \mathcal{E}_\eta) = F_*(\text{pr}_2^* \mathcal{E}_\eta) \quad (5.3)$$

(observe that F is a finite morphism, so $F_! = F_*$).

Proof. As remarked earlier, the Lang map $L_q(g) = \text{Fr}_q(g)g^{-1}$ identifies G with the quotient of G via the action of $G(\mathbf{F}_q)$ by right multiplication. Hence we can consider the intermediate quotient $\tilde{G} := G/H(\mathbf{F}_q)$, and we obtain finite étale morphisms

$$G \xrightarrow{\alpha} \tilde{G} \xrightarrow{\beta} G$$

where α is the quotient map and the composition equals L_q . Now α makes G an $H(\mathbf{F}_q)$ -torsor over \tilde{G} , so we can consider the $\overline{\mathbb{Q}}_\ell$ -local

³If G is unipotent, such a section always exists.

system on \tilde{G} associated to α via η . Let us denote this local system by $\tilde{\mathcal{E}}_\eta$. One can easily check that

$$\mathcal{E}_\rho \cong \beta_!(\tilde{\mathcal{E}}_\eta) = \beta_*(\tilde{\mathcal{E}}_\eta). \quad (5.4)$$

Now the map $\varphi_1 : (G/H) \times H \longrightarrow G$ defined by $(x, h) \mapsto s(x) \cdot h$ is an isomorphism. Moreover, φ_1 is compatible with the action of $H(\mathbf{F}_q)$ by right multiplication. Since $H/H(\mathbf{F}_q)$ is identified with H via the Lang map L_q for H , we obtain a commutative diagram in which the bottom row is an isomorphism

$$\begin{array}{ccc} (G/H) \times H & \xrightarrow{\varphi_1} & G \\ \text{id} \times L_q \downarrow & & \downarrow \alpha \\ (G/H) \times H & \xrightarrow{\cong} & \tilde{G} \end{array} \quad (5.5)$$

Let $\varphi_2 : (G/H) \times H \xrightarrow{\cong} \tilde{G}$ denote the bottom map in this diagram. Then for all $(x, h) \in (G/H) \times H$, we have $\beta \circ \alpha \circ \varphi_1(x, h) = L_q(s(x)h) = \text{Fr}_q(s(x)) \cdot L_q(h) \cdot s(x)^{-1}$, which means that $\beta \circ \varphi_2 = F$, where $F : (G/H) \times H \longrightarrow G$ is defined as in the statement of the proposition. (Indeed, both sides of the last equality give the same result when composed with $\text{id} \times L_q$ on the right, and $\text{id} \times L_q$ is an epimorphism.)

Next, looking at (5.5), we see that under the isomorphism φ_2 , the local system $\tilde{\mathcal{E}}_\eta$ on \tilde{G} corresponds to the local system $\text{pr}_2^* \mathcal{E}_\eta$ on $(G/H) \times H$. Finally, (5.4) implies that $\mathcal{E}_\rho \cong F_!(\text{pr}_2^* \mathcal{E}_\eta)$, completing the proof of the proposition. \square

Corollary 5.17. *In the situation of Proposition 5.16, we have*

$$t_{\mathcal{E}_\rho} = \text{ind}_{H(\mathbf{F}_q)}^{G(\mathbf{F}_q)} t_{\mathcal{E}_\eta} \quad (5.6)$$

where $\text{ind}_{H(\mathbf{F}_q)}^{G(\mathbf{F}_q)}$ denotes the induction map from conjugation-invariant functions on $H(\mathbf{F}_q)$ to conjugation-invariant functions on $G(\mathbf{F}_q)$.

Remark 5.18. In general, $t_{\mathcal{E}_\rho}$ is not equal to the character of the representation ρ , so formula (5.6) is not evident. On the other hand, by [Del77, Lem. 1.24 in *Sommes trig.*], $t_{\mathcal{E}_\eta}$ is a conjugation-invariant function on $H(\mathbf{F}_q)$, so formula (5.6) makes sense.

Proof of Corollary 5.17. By the Grothendieck-Lefschetz trace formula and (5.3),

$$t_{\mathcal{E}_\rho}(g) = \sum_{\substack{(x,h) \in (G/H)(\mathbf{F}_q) \times H(\mathbf{F}_q) \\ F(x,h)=g}} t_{\mathcal{E}_\eta}(h) \quad \forall g \in G(\mathbf{F}_q).$$

Now if $x \in (G/H)(\mathbf{F}_q)$, then $F(x, h) = s(x)hs(x)^{-1}$ for all h . Moreover, since G and H are both connected, we obtain $G(\mathbf{F}_q)/H(\mathbf{F}_q) \xrightarrow{\cong} (G/H)(\mathbf{F}_q)$, so as x ranges over $(G/H)(\mathbf{F}_q)$, we see that $s(x)$ ranges over a set of representatives of the left cosets of $H(\mathbf{F}_q)$ in $G(\mathbf{F}_q)$. Recalling the definition of the map $\text{ind}_{H(\mathbf{F}_q)}^{G(\mathbf{F}_q)}$, we obtain (5.6). \square

6 Properties of the reduced norm map

6.1 Summary

In this section we extend the reduced norm map $\text{Nm}^{n,q} : U^{n,q}(\mathbf{F}_Q) \rightarrow \mathbf{F}_q$ defined in Remark 4.2 to a morphism of \mathbf{F}_q -varieties $N^{n,q} : U^{n,q} \rightarrow \mathbb{G}_a$ and establish some properties of $N^{n,q}$.

To this end, let T be a formal indeterminate that commutes with all scalars, and given $g = 1 + a_1\tau + \cdots + a_n\tau^n \in U^{n,q}$, introduce the matrix $B(T) = (b_{ij}(T))_{i,j=1}^n$ whose entries are given by the following formula:

$$b_{ij}(T) = \begin{cases} a_{j-i}^{q^{i-1}} & \text{if } i < j; \\ 1 + a_n^{q^{i-1}} \cdot T & \text{if } i = j; \\ a_{n+j-i}^{q^{i-1}} \cdot T & \text{if } i > j. \end{cases}$$

We define $N^{n,q}(g) \in \mathbb{G}_a$ to be the coefficient of T in the polynomial $\det B(T)$. This determines a morphism of \mathbf{F}_q -varieties $N^{n,q} : U^{n,q} \rightarrow \mathbb{G}_a$.

Lemma 6.1. *If $g \in U^{n,q}(\mathbf{F}_Q)$, then $N^{n,q}(g) = \text{Nm}^{n,q}(g) \in \mathbf{F}_q \subset \mathbf{F}_Q$.*

This lemma is proved in §6.2.

Proposition 6.2. *We have $\text{pr}_n(L_Q(g)) = N^{n,q}(g)^q - N^{n,q}(g)$ for all $g \in U^{n,q}$.*

This proposition is proved in §6.3.

Remark 6.3. The formula $\text{pr}_n(L_Q(g)) = N^{n,q}(g)^q - N^{n,q}(g)$ together with the equality $N^{n,q}(1) = 0$ characterize the morphism $N^{n,q} : U^{n,q} \rightarrow \mathbb{G}_a$ uniquely. Indeed, let $N_1 : U^{n,q} \rightarrow \mathbb{G}_a$ be another morphism with the same properties. Then $(N^{n,q}(g) - N_1(g))^q = N^{n,q}(g) - N_1(g)$ for all $g \in U^{n,q}$, so the morphism $N^{n,q} - N_1 : U^{n,q} \rightarrow \mathbb{G}_a$ takes values in the discrete subset $\mathbf{F}_q \subset \mathbb{G}_a$. Since $U^{n,q}$ is connected, $N^{n,q} - N_1$ is constant, and since $N^{n,q}(1) = 0 = N_1(1)$ by assumption, we see that $N^{n,q} \equiv N_1$.

Corollary 6.4. $N^{n,q}(gh) = N^{n,q}(g) + N^{n,q}(h)$ for all $g \in U^{n,q}$, $h \in U^{n,q}(\mathbf{F}_Q)$.

Proof. Fix $h \in U^{n,q}(\mathbf{F}_Q)$. Then $L_Q(gh) = L_Q(g)$ for all $g \in U^{n,q}$, whence

$$N^{n,q}(gh)^q - N^{n,q}(gh) = N^{n,q}(g)^q - N^{n,q}(g)$$

by Proposition 6.2. This means that the \mathbf{F}_Q -morphism $U^{n,q} \otimes_{\mathbf{F}_q} \mathbf{F}_Q \rightarrow \mathbb{G}_a$ given by $g \mapsto N^{n,q}(gh) - N^{n,q}(g)$ takes values in the finite discrete subvariety $\mathbf{F}_q \subset \mathbb{G}_a$. Since $U^{n,q}$ is connected, this morphism is constant. Its value at $g = 1$ equals $N^{n,q}(h) - N^{n,q}(1) = N^{n,q}(h)$, which yields the corollary. \square

Corollary 6.5. If $g \in U^{n,q}(\overline{\mathbf{F}}_Q)$ is such that $\text{pr}_n(L_Q(g)) \in \mathbf{F}_Q$, then

$$\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}(\text{pr}_n(L_Q(g))) = N^{n,q}(\text{Fr}_Q(g)) - N^{n,q}(g).$$

Proof. By Proposition 6.2, $\text{pr}_n(L_Q(g)) = N^{n,q}(g)^q - N^{n,q}(g)$, whence

$$\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}(\text{pr}_n(L_Q(g))) = \sum_{i=0}^{n-1} \text{pr}_n(L_Q(g))^{q^i} = N^{n,q}(g)^Q - N^{n,q}(g).$$

But $N^{n,q}(g)^Q = \text{Fr}_Q(N^{n,q}(g)) = N^{n,q}(\text{Fr}_Q(g))$, completing the proof. \square

6.2 Proof of Lemma 6.1

We use the notation of Remark 4.2. Thus K is a local nonarchimedean field with residue field \mathbf{F}_q and L is the unramified extension of K of degree n . We let D be the central division algebra over K of invariant $1/n$ and choose the explicit presentation $D = L\langle\varpi\rangle/(\varpi^n - \pi)$, where $\pi \in K$ is a chosen uniformizer and $\varpi \cdot x = \text{Fr}(x) \cdot \varpi$ for all $x \in L$, with $\text{Fr} \in \text{Gal}(L/K)$ being the Frobenius (corresponding to the map $a \mapsto a^q$ on the residue field extension).

The division algebra D splits over L . Explicitly, one such splitting is given by the K -algebra homomorphism $\iota : D \hookrightarrow M_n(L)$ (where $M_n(L)$ is the associative algebra of n -by- n matrices over L) that takes every $a \in L$ to the diagonal matrix with entries $a, \text{Fr}(a), \dots, \text{Fr}^{n-1}(a)$ on the diagonal and takes $\varpi \in D$ to the matrix with 1's on the super-diagonal, π in the lower left corner and 0's elsewhere.

By definition, for any $x \in D$, the reduced norm of x is the determinant of the matrix $\iota(x)$ (it automatically belongs to K). Now if $a_1, \dots, a_n \in L$, then

$$\iota(1 + a_1\varpi + a_2\varpi^2 + \dots + a_n\varpi^n) = B(\pi), \quad (6.1)$$

where $B(T)$ is the matrix defined before the statement of Lemma 6.1.

Now assume that $a_j \in \mathcal{O}_L$ for all j and write \bar{a}_j for the image of a_j in the residue field of L , which we identify with \mathbf{F}_Q . If $g = 1 + \bar{a}_1\tau + \dots + \bar{a}_n\tau^n \in U^{n,q}(\mathbf{F}_Q)$, then

$$\det B(T) = 1 + N^{n,q}(g) \cdot T + O(T^2).$$

Using (6.1) and the definition of the reduced norm map $\text{Nm}^{n,q}$ given in Remark 4.2, we obtain $N^{n,q}(g) = \text{Nm}^{n,q}(g)$, proving Lemma 6.1.

6.3 Proof of Proposition 6.2

In principle, it is possible to prove Proposition 6.2 via a “brute force” calculation. However, the formulas one has to work with are quite complicated and the argument becomes difficult to follow. Therefore we prefer a roundabout approach, which is based on the following result, proved in §6.4.

Proposition 6.6. (a) *There is a unique morphism $N_1 : U^{n,q} \rightarrow \mathbb{G}_a$ of varieties over \mathbf{F}_q such that $N_1(1) = 0$ and $p_n(L_Q(g)) = N_1(g)^q - N_1(g)$ for all $g \in U^{n,q}$.*

(b) *If $g \in U^{n,q}$ and $h \in U^{n,q}(\mathbf{F}_Q)$, then $N_1(gh) = N_1(g) + N_1(h)$.*

(c) *We have $N_1(g) = \text{Nm}^{n,q}(g)$ for all $g \in U^{n,q}(\mathbf{F}_Q)$.*

To prove Proposition 6.2, we must show that if N_1 is the morphism defined in Proposition 6.6(a), then $N^{n,q} = N_1$. Consider $g = 1 + \sum_{j=1}^n a_j\tau^j \in U^{n,q}$ and view $N^{n,q}(g)$ and $N_1(g)$ as polynomial functions of (a_1, \dots, a_n) with coefficients in \mathbf{F}_q . Lemma 6.1 and Proposition 6.6(c) imply that these two functions agree on \mathbf{F}_Q^n . Hence if we show that each of these polynomials has total degree $< Q$, the proof will be complete.

Now by inspecting the definition of $N^{n,q}(g)$ via the determinant of the matrix $B(T)$ given in §6.1, we see that the total degree of $N^{n,q}(g)$ with respect to the variables a_1, \dots, a_n is at most $1 + q + q^2 + \dots + q^{n-1} = \frac{Q-1}{q-1} < Q$.

On the other hand, putting $x = \sum_{j=1}^n a_j \tau^j$ and expanding $g^{-1} = 1 - x + x^2 - \dots$, we find that the total degree of $\text{pr}_n(L_Q(g))$ with respect to the variables a_1, \dots, a_n is at most $Q + q \cdot (1 + q + q^2 + \dots + q^{n-1}) = q \cdot \frac{Q-1}{q-1}$, which implies that the total degree of $N_1(g)$ with respect to the variables a_1, \dots, a_n is at most $\frac{Q-1}{q-1} < Q$, completing the proof of Proposition 6.2.

6.4 Proof of Proposition 6.6

The uniqueness of N_1 follows from Remark 6.3, while (b) follows formally from (a) using the same argument as in the proof of Corollary 6.4. It remains to prove the existence of N_1 as well as assertion (c) of the proposition. To that end, we use the following

Lemma 6.7. *Every element of $U^{n,q}$ can be written uniquely as*

$$1 + a_1\tau + a_2\tau^2 + \dots + a_n\tau^n = (1 - b_1\tau) \cdot (1 - b_2\tau^2) \cdot \dots \cdot (1 - b_n\tau^n).$$

The maps relating each of the n -tuples (a_i) and (b_j) to the other one are polynomial maps with coefficients in \mathbf{F}_p .

Proof. This is completely straightforward: first observe that b_1 must necessarily equal $-a_1$. Then multiply both sides of the identity above by $(1 + a_1\tau)^{-1}$ on the left, and observe that the left hand side takes the form $1 + a'_2\tau^2 + \dots + a'_n\tau^n$, where the a'_j are certain polynomial functions of the a_i 's. Proceed by induction. \square

To prove the existence of N_1 using Lemma 6.7, we consider the following situation. Assume we are given an element of $U^{n,q}$ of the form

$$g = (1 - b_k\tau^k) \cdot (1 - b_{k+1}\tau^{k+1}) \cdot \dots \cdot (1 - b_n\tau^n),$$

where $1 \leq k \leq n$. We would like to show that there exists a polynomial map F_k (depending only on k) such that $p_n(L_Q(g)) = F_k(b_k, \dots, b_n)^q - F_k(b_k, \dots, b_n)$.

To this end, we use descending induction on k . When $k = n$, we have $g = 1 - b_n\tau^n$, so $L_Q(g) = 1 + (b_n - b_n^Q)\tau$, and we can take $F_n(b_n) = -(1 + b_n^q + b_n^{q^2} + \dots + b_n^{q^{n-1}})$.

Now suppose that $1 \leq k < n$ is arbitrary. We have

$$L_Q(g) = (1 - b_k^Q \tau^k) \cdot \dots \cdot (1 - b_n^Q \tau^n) \cdot (1 - b_n \tau^n)^{-1} \cdot \dots \cdot (1 - b_k \tau^k)^{-1},$$

which can be rewritten as

$$L_Q(g) = (1 - b_k^Q \tau^k) \cdot \left(1 + \sum_{i=k+1}^n c_i \tau^i\right) \cdot \left(1 + (b_k \tau^k) + (b_k \tau^k)^2 + \dots\right).$$

Here each c_i is some polynomial function of the variables b_{k+1}, \dots, b_n . Further, by induction, we may assume that

$$c_n = F_{k+1}(b_{k+1}, \dots, b_n)^q - F_{k+1}(b_{k+1}, \dots, b_n)$$

for some polynomial function F_{k+1} .

Expanding out the product above and collecting only the terms of degree n with respect to τ , we obtain the following expression:

$$\begin{aligned} c_n \tau^n &+ \left[(c_{n-k} \tau^{n-k}) \cdot (b_k \tau^k) + (c_{n-2k} \tau^{n-2k}) \cdot (b_k \tau^k)^2 + \dots \right] \\ &- (b_k^Q \tau^k) \cdot \left[(c_{n-k} \tau^{n-k}) + (c_{n-2k} \tau^{n-2k}) \cdot (b_k \tau^k) + \dots \right]. \end{aligned}$$

Thanks to our induction assumption, the term $c_n \tau^n$ can be ignored for the purpose of the present proof. The remaining terms can be regrouped as follows:

$$\sum_{i \geq 1} \left[(c_{n-ik} \tau^{n-ik}) \cdot (b_k \tau^k)^i - (b_k^Q \tau^k) \cdot (c_{n-ik} \tau^{n-ik}) \cdot (b_k \tau^k)^{i-1} \right].$$

It remains to observe that

$$\begin{aligned} &(c_{n-ik} \tau^{n-ik}) \cdot (b_k \tau^k)^i - (b_k^Q \tau^k) \cdot (c_{n-ik} \tau^{n-ik}) \cdot (b_k \tau^k)^{i-1} = \\ &= \left(c_{n-ik} \cdot b_k^{q^{n-ik}(1+q^k+\dots+q^{ki-k})} - c_{n-ik}^{q^k} \cdot b_k^{Q+q^{n-ik+k}(1+q^k+\dots+q^{ki-2k})} \right) \cdot \tau^n \\ &= \left(c_{n-ik} \cdot b_k^{q^{n-ik}+q^{n-ik+k}+\dots+q^{n-k}} - c_{n-ik}^{q^k} \cdot b_k^{q^{n-ik+k}+q^{n-ik+2k}+\dots+q^n} \right) \cdot \tau^n, \end{aligned}$$

and since

$$c_{n-ik} \cdot b_k^{q^{n-ik}+q^{n-ik+k}+\dots+q^{n-k}} - c_{n-ik}^{q^k} \cdot b_k^{q^{n-ik+k}+q^{n-ik+2k}+\dots+q^n} = A - A^{q^k}$$

where $A = c_{n-ik} \cdot b_k^{q^{n-ik}+q^{n-ik+k}+\dots+q^{n-k}}$, the induction step is complete.

Finally, define $N_1 : U^{n,q} \longrightarrow \mathbb{G}_a$ by the formula

$$N_1((1 - b_1\tau) \cdot (1 - b_2\tau^2) \cdot \dots \cdot (1 - b_n\tau^n)) = F_1(b_1, \dots, b_n).$$

From the proof above it is clear that N_1 has all the desired properties.

It remains to prove Proposition 6.6(c). We use an observation due to V. Drinfeld. Consider the action of the multiplicative group \mathbb{G}_m on $U^{n,q}$ given by (cf. §3.3)

$$\mathbb{G}_m \ni \lambda : g = 1 + \sum_{j=1}^n a_j \tau^j \longmapsto \lambda g \lambda^{-1} := 1 + \sum_{j=1}^n \lambda^{1-q^j} a_j \tau^j \quad (6.2)$$

Lemma 6.8. *The reduced norm map $\text{Nm}^{n,q} : U^{n,q}(\mathbf{F}_Q) \longrightarrow \mathbf{F}_q$ is the unique group homomorphism which is invariant under the action of \mathbf{F}_Q^\times on $U^{n,q}(\mathbf{F}_Q)$ coming from (6.2) and restricts to $\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}$ on the center $Z(\mathbf{F}_Q) = \mathbf{F}_Q$ of $U^{n,q}(\mathbf{F}_Q)$.*

Proof. The fact that $\text{Nm}^{n,q}$ has all of the stated properties follows easily from its definition given in Remark 4.2. To check the uniqueness assertion, let $H \subset U^{n,q}(\mathbf{F}_Q)$ be the subgroup generated by all elements of the form $g^{-1} \cdot (\lambda g \lambda^{-1})$ with $g \in U^{n,q}(\mathbf{F}_Q)$ and $\lambda \in \mathbf{F}_Q^\times$. It suffices to show that $U^{n,q}(\mathbf{F}_Q) = H \cdot Z(\mathbf{F}_Q)$.

Assume that this is not the case, and let $g = 1 + \sum_{j=k}^n a_j \tau^j \in U^{n,q}(\mathbf{F}_Q)$ be an element that does not belong to $H \cdot Z(\mathbf{F}_Q)$, where $k \geq 1$ is as large as possible. In particular, $k < n$. Hence there exists $\lambda \in \mathbf{F}_Q^\times$ with $\lambda^{1-q^k} \neq 1$. Put $b = \frac{a_k}{\lambda^{1-q^k} - 1}$ and $g_1 = 1 + b\tau^k$. Then $g_1^{-1} \cdot (\lambda g_1 \lambda^{-1}) = 1 + a_k \tau^k + O(\tau^{k+1})$, whence $g = g_1^{-1} \cdot (\lambda g_1 \lambda^{-1}) \cdot g'$ for some $g' \in U^{n,q}(\mathbf{F}_Q)$ such that $g' = 1 + O(\tau^{k+1})$. The maximality of k implies that $g' \in H \cdot Z(\mathbf{F}_Q)$, which is a contradiction. \square

To see that Lemma 6.8 implies Proposition 6.6(c), we argue as follows. As a special case of Proposition 6.6(b), which was already proved, we see that $N_1 : U^{n,q}(\mathbf{F}_Q) \rightarrow \mathbf{F}_q$ is a group homomorphism. Hence it suffices to check that N_1 is invariant under the action (6.2) and that $N_1(1 + a\tau^n) = \text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}(a)$ for all $a \in \mathbf{F}_Q$.

Choose any $1 \leq j \leq n$, pick $x \in \mathbf{F}_Q$, and consider $g = 1 - x\tau^j \in U^{n,q}(\mathbf{F}_Q)$. We will check that $N_1(\lambda g \lambda^{-1}) = N_1(g)$ for any $\lambda \in \mathbf{F}_Q^\times$, and, in addition, if $j = n$, then $N_1(x) = -\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}(x)$. Since (6.2) is a group action, and since $N_1 : U^{n,q}(\mathbf{F}_Q) \longrightarrow \mathbf{F}_q$ is a homomorphism, it will follow from Lemma 6.7 that $N_1 : U^{n,q}(\mathbf{F}_Q) \longrightarrow \mathbf{F}_q$ is invariant under the \mathbf{F}_Q^\times -action coming from (6.2) and the proof will be complete.

Now consider the restriction of N_1 to the subvariety of $U^{n,q}$ consisting of all points of the form $1 - b\tau^j$ (this subvariety is isomorphic to \mathbb{A}^1). We can calculate this restriction explicitly as follows. We have

$$L_Q(1 - b\tau^j) = (1 - b^{q^n}\tau^j) \cdot [1 + (b\tau^j) + (b\tau^j)^2 + \cdots],$$

so if $j \nmid n$, we get $N_1(1 - b\tau^j) = 0$, and if $j \mid n$, we get

$$p_n(L_Q(1 - b\tau^j)) = b^{1+q^j+q^{2j}+\cdots+q^{n-j}} - b^{q^j+q^{2j}+\cdots+q^n},$$

whence⁴

$$N_1(1 - b\tau^j) = -\varphi(b) - \varphi(b)^q - \varphi(b)^{q^2} - \cdots - \varphi(b)^{q^{j-1}},$$

where $\varphi(b) := b^{1+q^j+\cdots+q^{n-j}}$. In particular, if $j = n$, we obtain $\varphi(x) = x$ and $N_1(1 - x\tau^n) = -\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_q}(x)$. In addition, if j is arbitrary, then given $\lambda \in \mathbf{F}_Q^\times$, we have $\lambda(1 - b\tau^j)\lambda^{-1} = 1 - \lambda^{1-q^j}b\tau^j$. So if $j \nmid n$, we get $N_1(\lambda(1 - b\tau^j)\lambda^{-1}) = 0 = N_1(1 - b\tau^j)$. If $j \mid n$, then with the notation above, $\varphi(\lambda^{1-q^j}b) = \lambda^{1-q^n}\varphi(b) = \varphi(b)$, so we again have $N_1(\lambda(1 - b\tau^j)\lambda^{-1}) = N_1(1 - b\tau^j)$, completing the proof.

7 Proof of the main theorem

7.1 Outline of the argument

We begin by describing step-by-step the strategy that will lead to our proof of Theorem 4.4. Throughout this section we fix a character $\psi : \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}}_\ell^\times$ and let q^m be its conductor. We also write $n_1 = n/m$ and $q_1 = q^m$, and let $\psi_1 : \mathbf{F}_{q_1} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the character such that $\psi = \psi_1 \circ \text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q_1}}$.

We denote by $\text{pr}_n : U^{n,q} \rightarrow \mathbb{G}_a$ the projection onto the last factor:

$$\text{pr}_n(1 + a_1\tau + \cdots + a_n\tau^n) = a_n.$$

If $W \subset U^{n,q}$ is a subvariety, we also write pr_n for the restriction of pr_n to W .

⁴Here we are implicitly using the fact that \mathbb{A}^1 is connected to ensure that the expression we wrote down coincides with $N_1(1 - b\tau^j)$ for all $b \in \mathbb{A}^1$.

Step 1

We first obtain some information about the irreducible representations of the group $U^{n,q}(\mathbf{F}_Q)$. To this end, along with the closed connected subgroup

$$H_m = \left\{ 1 + \sum_{\substack{j \leq n/2 \\ m|j}} a_j \tau^j + \sum_{n/2 < j \leq n} a_j \tau^j \right\} \subset U^{n,q}$$

defined in Theorem 4.4(c), we introduce two more:

$$H_m^+ = \left\{ 1 + \sum_{\substack{j < n/2 \\ m|j}} a_j \tau^j + \sum_{n/2 \leq j \leq n} a_j \tau^j \right\} \subset U^{n,q}$$

and

$$H_m^- = \left\{ 1 + \sum_{\substack{n/2 < j < n \\ m \nmid j}} a_j \tau^j + a_n \tau^n \right\} \subset U^{n,q}.$$

By construction, $H_m^- \subset H_m \subset H_m^+$. The subgroup H_m^- will play a role in the other steps of the proof as well.

Remarks 7.1. (1) We have $H_m^+ = H_m$ unless m is even and n_1 is odd, in which case H_m is a normal subgroup of H_m^+ of codimension 1.

(2) If $m = n$, then $H_m^- = H_m$.

The following lemma is proved in §7.2.

Lemma 7.2. *If ρ is any irreducible representation of $U^{n,q}(\mathbf{F}_Q)$ with central character ψ , then the restriction of ρ to $H_m^-(\mathbf{F}_Q)$ contains the 1-dimensional representation $\psi \circ \text{pr}_n : H_m^-(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$.*

Now consider the character⁵ $\tilde{\psi} := \psi_1 \circ \text{Nm}^{n_1, q_1} \circ \nu_m : H_m(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$.

Remark 7.3. Recall that $\nu_m : H_m(\mathbf{F}_Q) \rightarrow U^{n_1, q_1}(\mathbf{F}_Q)$ is the map that discards all summands $a_j \tau^j$ with $m \nmid j$ (cf. Remark 4.3). Hence $\tilde{\psi}|_{H_m^-(\mathbf{F}_Q)} = \psi \circ \text{pr}_n$.

⁵The fact that the projection map $\nu_m : H_m \rightarrow U^{n_1, q_1}$ is a group homomorphism is verified by a direct calculation, and since the restriction of $\text{Nm}^{n_1, q_1} : U^{n_1, q_1}(\mathbf{F}_Q) \rightarrow \mathbf{F}_{q_1}$ to $Z(\mathbf{F}_Q)$ is equal to $\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q_1}} : \mathbf{F}_Q \rightarrow \mathbf{F}_{q_1}$, we see that $\tilde{\psi}$ is indeed a character that extends $\psi : Z(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$.

Proposition 7.4. (a) Suppose m is odd or n_1 is even. Then

$$\rho_\psi := \text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$$

is an irreducible representation of $U^{n,q}(\mathbf{F}_Q)$.

(b) Suppose m is even and n_1 is odd⁶. Let $\Gamma_m \subset U^{n,q}(\mathbf{F}_Q)$ be the subgroup defined in Theorem 4.4(c); in other words,

$$\Gamma_m = \left\{ \gamma = 1 + \sum_{j=1}^n a_j \tau^j \mid \gamma \in H_m^+(\mathbf{F}_Q) \text{ and } a_{n/2} \in \mathbf{F}_{q^{n/2}} \right\}.$$

Then $\tilde{\psi}$ can be extended to a character of Γ_m , and if $\chi : \Gamma_m \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is any such extension, then $\rho_\psi := \text{Ind}_{\Gamma_m}^{U^{n,q}(\mathbf{F}_Q)}(\chi)$ is an irreducible representation of $U^{n,q}(\mathbf{F}_Q)$, which is independent of the choice of χ . Furthermore, $\text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$ is isomorphic to a direct sum of $q^{n/2}$ copies of ρ_ψ .

In both cases, the restriction of ρ_ψ to $H_m^-(\mathbf{F}_Q)$ contains $\psi \circ \text{pr}_n$; in particular, ρ_ψ has central character ψ .

This proposition is proved in §7.3.

Step 2

We consider $H_c^\bullet(X, \overline{\mathbb{Q}_\ell}) = \bigoplus_{i \in \mathbb{Z}} H_c^i(X, \overline{\mathbb{Q}_\ell})$ as a finite dimensional graded vector space over $\overline{\mathbb{Q}_\ell}$ equipped with commuting actions of $U^{n,q}(\mathbf{F}_Q)$ and Fr_Q . In particular, given any representation (not necessarily irreducible) ξ of $U^{n,q}(\mathbf{F}_Q)$, we obtain a graded vector space $\text{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\xi, H_c^\bullet(X, \overline{\mathbb{Q}_\ell}))$ with an action of Fr_Q .

Now consider the representation $\xi_\psi = \text{Ind}_{H_m^-(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\psi \circ \text{pr}_n)$. In view of Lemma 7.2, ξ_ψ is isomorphic to a direct sum of all irreducible representations of $U^{n,q}(\mathbf{F}_Q)$ that have central character ψ , taken with certain multiplicities.

Proposition 7.5. $\text{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\xi_\psi, H_c^\bullet(X, \overline{\mathbb{Q}_\ell}))$ is concentrated in degree $n + n_1 - 2$. It has dimension 1 if m is odd or n_1 is even, and it has dimension $q^{n/2}$ if m is even and n_1 is odd. Fr_Q acts on it via the scalar $(-1)^{n-n_1} \cdot q^{n(n+n_1-2)/2}$.

This proposition is proved in §7.4.

⁶Equivalently, n is even and m does not divide $n/2$.

Step 3

The last ingredient is the following result, proved in §7.6.

Proposition 7.6. *If ρ_ψ is the representation of $U^{n,q}(\mathbf{F}_Q)$ constructed in Proposition 7.4, then $\mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\rho_\psi, H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)) \neq 0$.*

The finale

Let us show that combining the three steps above, we obtain a proof of Theorem 4.4. Write ρ_ψ for the irreducible representation of $U^{n,q}(\mathbf{F}_Q)$ constructed in Proposition 7.4. Then $H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)$ contains ρ_ψ as a direct summand by Proposition 7.6. Introduce the following multiplicities:

$$d_1 = \dim \mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\rho_\psi, H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)) \geq 1,$$

$$d_2 = \dim \mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\rho_\psi, \xi_\psi),$$

$$d_3 = \dim \mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\xi_\psi, H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)).$$

Then d_2 is at least the multiplicity of ρ_ψ in $\mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$. Furthermore, it is clear that $d_3 \geq d_1 \cdot d_2$, and equality holds if and only if ρ_ψ is the *unique* irreducible representation of $U^{n,q}(\mathbf{F}_Q)$ that appears both in $H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)$ and in ξ_ψ .

We now claim that $d_2 \geq d_3$. Indeed, combining Proposition 7.4 with Proposition 7.5, we see that

$$d_3 = \dim \mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\rho_\psi, \mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})).$$

The last assertion of Remark 7.3 implies that $\mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$ is a direct summand of ξ_ψ , whence $d_2 \geq d_3$.

Comparing the assertions of the last two paragraphs, we find that $d_1 = 1$ and $d_1 \cdot d_2 = d_2 = d_3$. In view of the first and third assertions of Proposition 7.5, we see that all parts of Theorem 4.4 follow.

7.2 Proof of Lemma 7.2

Fix an irreducible representation ρ of $U^{n,q}(\mathbf{F}_Q)$ with central character ψ . We must prove that the restriction of ρ to $H_m^-(\mathbf{F}_Q)$ contains the 1-dimensional representation $\psi \circ \mathrm{pr}_n : H_m^-(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$. For each integer k we write $U^{\geq k} \subset U^{n,q}$ for the subgroup consisting of elements

of the form $1 + \sum_{j=k}^n a_j \tau^j$. We will show, using descending induction on k , that for any k the restriction of ρ to $H_m^-(\mathbf{F}_Q) \cap U^{\geq k}(\mathbf{F}_Q)$ contains the character $\psi \circ \text{pr}_n$. This will imply the lemma.

If $k = n$, there is nothing to prove. So we assume that $k < n$ and that the assertion in the previous paragraph holds for $k+1$ in place of k . Further, we may assume that $k > n/2$ and $m \nmid k$, since otherwise $H_m^-(\mathbf{F}_Q) \cap U^{\geq k}(\mathbf{F}_Q) = H_m^-(\mathbf{F}_Q) \cap U^{\geq k+1}(\mathbf{F}_Q)$.

By the induction hypothesis, the restriction of ρ to $H_m^-(\mathbf{F}_Q) \cap U^{\geq k+1}(\mathbf{F}_Q)$ contains $\psi \circ \text{pr}_n$. This implies that the restriction of ρ to $H_m^-(\mathbf{F}_Q) \cap U^{\geq k}(\mathbf{F}_Q)$ contains some character $\chi : H_m^-(\mathbf{F}_Q) \cap U^{\geq k}(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that

$$\chi|_{H_m^-(\mathbf{F}_Q) \cap U^{\geq k+1}(\mathbf{F}_Q)} = \psi \circ \text{pr}_n.$$

The subgroup $U^{\geq n-k}(\mathbf{F}_Q) \subset U^{n,q}(\mathbf{F}_Q)$ normalizes $H_m^-(\mathbf{F}_Q) \cap U^{\geq k}(\mathbf{F}_Q)$ and centralizes $H_m^-(\mathbf{F}_Q) \cap U^{\geq k+1}(\mathbf{F}_Q)$. It will be enough to find an element $g \in U^{\geq n-k}(\mathbf{F}_Q)$ that conjugates χ into the character $\psi \circ \text{pr}_n$ on $H_m^-(\mathbf{F}_Q) \cap U^{\geq k}(\mathbf{F}_Q)$.

To this end, observe that by construction, we can write

$$\chi\left(1 + \sum_{\substack{k \leq j < n \\ m \nmid j}} a_j \tau^j + a_n \tau^n\right) = \chi_1(a_k) \cdot \psi(a_n)$$

for some character $\chi_1 : \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Let $g = 1 + a_{n-k} \tau^{n-k} \in U^{\geq n-k}(\mathbf{F}_Q)$, where $a_{n-k} \in \mathbf{F}_Q$ will be chosen later. Then a direct calculation shows that

$$\chi\left(g \cdot \left(1 + \sum_{\substack{k \leq j < n \\ m \nmid j}} a_j \tau^j + a_n \tau^n\right) \cdot g^{-1}\right) = \chi_1(a_k) \cdot \psi(a_n + a_{n-k} a_k^{q^{n-k}} - a_k a_{n-k}^{q^k}).$$

It remains to check that $a_{n-k} \in \mathbf{F}_Q$ can be chosen so that

$$\chi_1(x) = \psi(a_{n-k}^{q^k} x - a_{n-k} x^{q^{n-k}}) \quad (7.1)$$

for all $x \in \mathbf{F}_Q$. As explained in §5.4, there exist (uniquely determined) $y, b \in \mathbf{F}_Q$ such that $\chi_1(x) = \psi_0(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(yx))$ and $\psi(x) = \psi_0(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(bx))$ for all $x \in \mathbf{F}_Q$, where $\psi_0 : \mathbf{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is the chosen nontrivial character. Then

$$\begin{aligned} \psi(a_{n-k}^{q^k} x - a_{n-k} x^{q^{n-k}}) &= \psi_0\left(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(b a_{n-k}^{q^k} x - b a_{n-k} x^{q^{n-k}})\right) \\ &= \psi_0\left(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(a_{n-k}^{q^{-(n-k)}} \cdot x \cdot (b - b^{q^k}))\right), \end{aligned}$$

where we used the identities $a_{n-k}^{q^k} = a_{n-k}^{q^{-(n-k)}}$ and $b^{q^k} = b^{q^{-(n-k)}}$ (which hold because $a_{n-k}, b \in \mathbf{F}_Q = \mathbf{F}_{q^n}$) together with the fact that $\mathrm{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(z^{q^{n-k}}) = \mathrm{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(z)$ for all $z \in \mathbf{F}_Q$. Since $m \nmid k$ and ψ has conductor q^m by assumption, we have $b^{q^k} \neq b$. So if we choose $a_{n-k} = (y \cdot (b - b^{q^k}))^{q^{n-k}}$, then (7.1) is satisfied. \square

7.3 Proof of Proposition 7.4

Write $U^{>n/2} \subset U^{n,q}$ for the subgroup consisting of elements of the form $1 + \sum_{n/2 < j \leq n} a_j \tau^j$. This is a normal abelian subgroup of $U^{n,q}$, which is contained in H_m . We first establish the following

Lemma 7.7. *The normalizer in $U^{n,q}(\mathbf{F}_Q)$ of the character*

$$\tilde{\psi}|_{U^{>n/2}(\mathbf{F}_Q)} : U^{>n/2}(\mathbf{F}_Q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

is equal to $H_m^+(\mathbf{F}_Q)$.

Proof. First let us check that $H_m^+(\mathbf{F}_Q)$ does normalize $\tilde{\psi}|_{U^{>n/2}(\mathbf{F}_Q)}$. If m is odd or n_1 is even, then $H_m^+ = H_m$, so there is nothing to do. Suppose that m is even and n_1 is odd. It is enough to show that any element of the form $g = 1 + a_{n/2} \tau^{n/2} \in U^{n,q}(\mathbf{F}_Q)$ normalizes $\tilde{\psi}|_{U^{>n/2}(\mathbf{F}_Q)}$. But in fact, g centralizes $U^{>n/2}(\mathbf{F}_Q)$.

Now, to obtain a contradiction, assume that there exists an element $g \in U^{n,q}(\mathbf{F}_Q)$ such that $g \notin H_m^+(\mathbf{F}_Q)$ and g normalizes $\tilde{\psi}|_{U^{>n/2}(\mathbf{F}_Q)}$. Write $g = 1 + \sum_{j=1}^n a_j \tau^j$ and let k be the smallest integer such that $a_k \neq 0$ and $m \nmid k$. Then $k < n/2$. Multiplying g by a suitable element of $U^{n_1,q_1}(\mathbf{F}_Q)$ on the right, we may assume that $a_j = 0$ for all $1 \leq j < k$. Next consider the subgroup of $U^{>n/2}(\mathbf{F}_Q)$ consisting of all elements of the form $1 + c_{n-k} \tau^{n-k} + c_n \tau^n$. By assumption, we have

$$\begin{aligned} \psi(c_n) &= \tilde{\psi}(1 + c_{n-k} \tau^{n-k} + c_n \tau^n) \\ &= \tilde{\psi}(g \cdot (1 + c_{n-k} \tau^{n-k} + c_n \tau^n) \cdot g^{-1}) \\ &= \tilde{\psi}(1 + c_{n-k} \tau^{n-k} + (c_n + a_k c_{n-k}^{q^k} - c_{n-k} a_k^{q^{n-k}}) \tau^n) \\ &= \psi(c_n + a_k c_{n-k}^{q^k} - c_{n-k} a_k^{q^{n-k}}) = \psi(a_n) \cdot \psi(a_k c_{n-k}^{q^k} - c_{n-k} a_k^{q^{n-k}}). \end{aligned}$$

We see that $\psi(a_k c_{n-k}^{q^k} - c_{n-k} a_k^{q^{n-k}}) = 1$ for all $c_{n-k} \in \mathbf{F}_Q$. We claim that this is a contradiction. Indeed, as in §7.2, choose $b \in \mathbf{F}_Q$ with

$\psi(x) = \psi_0(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(bx))$ for all $x \in \mathbf{F}_Q$. Then the computation from §7.2 shows that

$$\psi(a_k c_{n-k}^{q^k} - c_{n-k} a_k^{q^{n-k}}) = \psi_0\left(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(c_{n-k}^{q^{-(n-k)}} \cdot a_k \cdot (b - b^{q^k}))\right),$$

and since $b^{q^k} \neq b$ and $a_k \neq 0$, the right hand side cannot be 1 for all $c_{n-k} \in \mathbf{F}_Q$. \square

We proceed with the proof of Proposition 7.4. If m is odd or n_1 is even, then by Lemma 7.7, the normalizer in $U^{n,q}(\mathbf{F}_Q)$ of $\tilde{\psi}|_{U^{>n/2}(\mathbf{F}_Q)}$ is equal to $H_m(\mathbf{F}_Q)$, so applying Lemma 5.1 to $\Gamma = U^{n,q}(\mathbf{F}_Q)$, $N = U^{>n/2}(\mathbf{F}_Q)$ and $\chi = \tilde{\psi}|_{U^{>n/2}(\mathbf{F}_Q)}$ implies that $\text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi})$ is irreducible, proving part (a) of the proposition.

Next assume that m is even and n_1 is odd. Let us apply Lemma 5.2 to the group $H = H_m^+(\mathbf{F}_Q)$, the normal subgroup $N = H_m(\mathbf{F}_Q)$ of H and the character $\chi = \tilde{\psi}$. By Lemma 7.7, $\tilde{\psi}$ is invariant under $H_m^+(\mathbf{F}_Q)$ -conjugation. The quotient $H_m^+(\mathbf{F}_Q)/H_m(\mathbf{F}_Q)$ can be naturally identified with the additive group of \mathbf{F}_Q . To calculate the induced “commutator pairing”

$$B_{\tilde{\psi}} : (H_m^+(\mathbf{F}_Q)/H_m(\mathbf{F}_Q)) \times (H_m^+(\mathbf{F}_Q)/H_m(\mathbf{F}_Q)) \longrightarrow \overline{\mathbb{Q}_\ell}^\times,$$

we observe that if $g = 1 + x\tau^{n/2}$ and $h = 1 + y\tau^{n/2}$ with $x, y \in \mathbf{F}_Q$, then

$$ghg^{-1}h^{-1} = 1 + (x \cdot y^{q^{n/2}} - y \cdot x^{q^{n/2}}) \cdot \tau^n,$$

whence $B_{\tilde{\psi}}$ can be identified with the pairing

$$\mathbf{F}_Q \times \mathbf{F}_Q \longrightarrow \overline{\mathbb{Q}_\ell}^\times, \quad (x, y) \longmapsto \psi(x \cdot y^{q^{n/2}} - y \cdot x^{q^{n/2}}). \quad (7.2)$$

Lemma 7.8. *The pairing (7.2) is nondegenerate, and the additive subgroup $\mathbf{F}_{q^{n/2}} \subset \mathbf{F}_Q = \mathbf{F}_{q^n}$ is maximal isotropic with respect to it.*

Proof. It is clear that $\mathbf{F}_{q^{n/2}}$ is isotropic with respect to (7.2). If we show that (7.2) is nondegenerate, then the maximality will follow from the fact that $\#\mathbf{F}_{q^{n/2}} = \sqrt{\#\mathbf{F}_Q}$. To this end, as in §7.2, choose $b \in \mathbf{F}_Q$ such that $\psi(x) = \psi_0(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(bx))$ for all $x \in \mathbf{F}_Q$. Assume that $y \in \mathbf{F}_Q$ is such that $\psi(x \cdot y^{q^{n/2}} - y \cdot x^{q^{n/2}}) = 1$ for all $x \in \mathbf{F}_Q$. Then $\psi_0\left(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(b \cdot x \cdot y^{q^{n/2}} - b^{q^{n/2}} \cdot y^{q^{n/2}} \cdot x)\right) = 1$ for all $x \in \mathbf{F}_Q$, where we used the identities $b^{q^{-n/2}} = b^{q^{n/2}}$, $y^{q^{-n/2}} = y^{q^{n/2}}$ and the

fact that $\mathrm{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(z^{q^{n/2}}) = \mathrm{Tr}_{\mathbf{F}_Q/\mathbf{F}_p}(z)$ for all $z \in \mathbf{F}_Q$. This forces $(b - b^{q^{n/2}}) \cdot y^{q^{n/2}} = 0$. Since ψ has conductor q^m and m does not divide $n/2$ by assumption, we have $b \neq b^{q^{n/2}}$, whence $y = 0$, as needed. \square

Now we complete the proof of Proposition 7.4(b). Note that the subgroup Γ_m equals the preimage of $\mathbf{F}_{q^{n/2}} \subset \mathbf{F}_Q = H_m^+(\mathbf{F}_Q)/H_m(\mathbf{F}_Q)$ in $H_m^+(\mathbf{F}_Q)$. By Lemmas 5.2 and 7.8, $\mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{H_m^+(\mathbf{F}_Q)}(\tilde{\psi})$ is a direct sum of $q^{n/2}$ copies of a single irreducible representation ρ of $H_m^+(\mathbf{F}_Q)$. Moreover, $\tilde{\psi}$ can be extended to a character of Γ_m , and if is any such extension, then $\rho \cong \mathrm{Ind}_{\Gamma_m}^{H_m^+(\mathbf{F}_Q)}(\chi)$. We see that

$$\rho_\psi := \mathrm{Ind}_{\Gamma_m}^{U^{n,q}(\mathbf{F}_Q)}(\chi) \cong \mathrm{Ind}_{H_m^+(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\rho)$$

is independent of the choice of χ , and

$$\mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi}) \cong \mathrm{Ind}_{H_m^+(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)} \mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{H_m^+(\mathbf{F}_Q)}(\tilde{\psi})$$

is isomorphic to a direct sum of $q^{n/2}$ copies of ρ_ψ , completing the proof.

7.4 Proof of Proposition 7.5

For this subsection and the next one, all varieties and algebraic groups we will consider are assumed to be defined over \mathbf{F}_Q . Thus we simply write $U^{n,q}$ and U^{n_1,q_1} in place of $U^{n,q} \otimes_{\mathbf{F}_q} \mathbf{F}_Q$ and $U^{n_1,q_1} \otimes_{\mathbf{F}_{q_1}} \mathbf{F}_Q$, respectively. As explained in Remark 4.3, we identify U^{n_1,q_1} with the subgroup of $U^{n,q}$ consisting of all elements of the form $1 + \sum_{m|j} a_j \tau^j$.

7.4.1 Preliminary considerations

By Corollary 5.13, we have

$$\mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}(\xi_\psi, H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)) \cong H_c^\bullet(Y, \mathcal{E}_{\xi_\psi}|_Y) \quad (7.3)$$

as graded vector spaces with an action of Fr_Q , where $Y = \mathrm{pr}_n^{-1}(0) \subset U^{n,q}$ and \mathcal{E}_{ξ_ψ} is the $\overline{\mathbb{Q}}_\ell$ -local system on $U^{n,q}$ associated to the Lang torsor $U^{n,q} \xrightarrow{L_Q} U^{n,q}$ via ξ_ψ .

We will calculate \mathcal{E}_{ξ_ψ} using Proposition 5.16. Observe that H_m^- is a connected commutative algebraic group, so if $\mathcal{E}_{\psi \circ \text{pr}_n}$ is the $\overline{\mathbb{Q}}_\ell$ -local system on H_m^- coming from the Lang isogeny $L_Q : H_m^- \rightarrow H_m^-$ and the character $\psi \circ \text{pr}_n : H_m^-(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, then by Lemma 5.14, $\mathcal{E}_{\psi \circ \text{pr}_n}$ is also the unique multiplicative local system on H_m^- corresponding to the character $\psi \circ \text{pr}_n$ as in Lemma 5.6. In fact,

$$\mathcal{E}_{\psi \circ \text{pr}_n} \cong \text{pr}_n^* \mathcal{L}_\psi,$$

where \mathcal{L}_ψ is the multiplicative local system on \mathbb{G}_a corresponding to $\psi : \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}}_\ell^\times$.

7.4.2 Auxiliary notation

Let I' denote the set of integers j such that $n/2 < j < n$ and $m \nmid j$. Put $I = I' \cup \{n\}$ and $J = \{1, 2, \dots, n\} \setminus I$. Then we can write

$$H_m^- = \left\{ 1 + \sum_{i \in I} a_i \tau^i \right\} \subset U^{n,q},$$

and we can identify $U^{n,q}/H_m^-$ with an affine space \mathbb{A}^d of dimension

$$d := \#J = \begin{cases} \frac{n+n_1}{2} - 1 & \text{if } m \text{ is odd or } n_1 \text{ is even,} \\ \frac{n+n_1+1}{2} - 1 & \text{if } m \text{ is even and } n_1 \text{ is odd.} \end{cases}$$

We will denote the coordinates of this affine space by $(a_j)_{j \in J}$.

7.4.3 A reformulation of Proposition 7.5

The morphism

$$s : \mathbb{A}^d \rightarrow U^{n,q}, \quad (a_j)_{j \in J} \mapsto 1 + \sum_{j \in J} a_j \tau^j$$

is a section of the quotient map $U^{n,q} \rightarrow U^{n,q}/H_m^-$. By Proposition 5.16,

$$\mathcal{E}_{\xi_\psi} \cong F_!(\tilde{\text{pr}}_n^* \mathcal{L}_\psi), \quad (7.4)$$

where $F : \mathbb{A}^d \times H_m^- \rightarrow U^{n,q}$ is given by $(x, h) \mapsto \text{Fr}_Q(s(x))hs(x)^{-1}$ and $\tilde{\text{pr}}_n : \mathbb{A}^d \times H_m^- \rightarrow \mathbb{G}_a$ is the composition of the second projection $\mathbb{A}^d \times H_m^- \rightarrow H_m^-$ with $\text{pr}_n : H_m^- \rightarrow \mathbb{G}_a$. By (7.4) and the proper base change theorem,

$$H_c^\bullet(Y, \mathcal{E}_{\xi_\psi}|_Y) \cong H_c^\bullet(F^{-1}(Y), \tilde{\text{pr}}_n^* \mathcal{L}_\psi|_{F^{-1}(Y)}). \quad (7.5)$$

Now note that if $h = 1 + \sum_{i \in I} a_i \tau^i \in H_m^-$ and we put $h^\circ := 1 + \sum_{i \in I \setminus \{n\}} a_i \tau^i$, then for any $g_1, g_2 \in U^{n,q}$, we have $\text{pr}_n(g_1 h g_2) = a_n + \text{pr}_n(g_1 h^\circ g_2)$. In particular, for any $x \in \mathbb{A}^d$, we have $\text{pr}_n(F(x, h)) = a_n + \text{pr}_n(F(x, h^\circ))$. This implies that the map

$$\mathbb{A}^d \times H_m^- \longrightarrow \mathbb{A}^d \times (H_m^- \cap Y), \quad (x, h) \longmapsto (x, h^\circ)$$

(which is the projection onto the first $n-1$ coordinates) yields an isomorphism between $F^{-1}(Y) \subset \mathbb{A}^d \times H_m^-$ and $\mathbb{A}^d \times (H_m^- \cap Y)$. Under this isomorphism, the local system $\tilde{\text{pr}}_n^* \mathcal{L}_\psi|_{F^{-1}(Y)}$ corresponds to the local system $\alpha^*(\mathcal{L}_\psi)$ on $\mathbb{A}^d \times (H_m^- \cap Y)$, where $\alpha : \mathbb{A}^d \times (H_m^- \cap Y) \longrightarrow \mathbb{G}_a$ is given by $\alpha(x, h) = -\text{pr}_n(\text{Fr}_Q(s(x)) h s(x)^{-1})$. So in view of (7.3) and (7.5), Proposition 7.5 is equivalent to the following assertions:

$$H_c^r(\mathbb{A}^d \times (H_m^- \cap Y), \alpha^*(\mathcal{L}_\psi)) = 0 \quad \text{if } r \neq n + n_1 - 2; \quad (7.6)$$

$$\begin{aligned} \dim H_c^{n+n_1-2}(\mathbb{A}^d \times (H_m^- \cap Y), \alpha^*(\mathcal{L}_\psi)) &= \\ &= \begin{cases} 1 & \text{if } m \text{ is odd or } n_1 \text{ is even,} \\ q^{n/2} & \text{if } m \text{ is even and } n_1 \text{ is odd;} \end{cases} \end{aligned} \quad (7.7)$$

$$\begin{aligned} \text{Fr}_Q \text{ acts on } \dim H_c^{n+n_1-2}(\mathbb{A}^d \times (H_m^- \cap Y), \alpha^*(\mathcal{L}_\psi)) \\ \text{as multiplication by the scalar } (-1)^{n-n_1} \cdot q^{n(n+n_1-2)/2}. \end{aligned} \quad (7.8)$$

7.4.4 Additional notation

There are $n-d-1$ integers j such that $n > j > n/2$ and $m \nmid j$. Let us label them as follows: $j_1 > j_2 > \dots > j_{n-d-1}$. Thus $I = \{n\} \cup I^\circ$, where $I^\circ = \{j_1, j_2, \dots, j_{n-d-1}\}$, and $J = \{m, 2m, \dots, (n_1-1)m\} \cup J^\circ$, where

$$J^\circ = \begin{cases} \{n - j_1, \dots, n - j_{n-d-1}\} & \text{if } m \text{ is odd or } n_1 \text{ is even,} \\ \{n - j_1, \dots, n - j_{n-d-1}, n/2\} & \text{if } m \text{ is even and } n_1 \text{ is odd.} \end{cases}$$

From now on we will identify $\mathbb{A}^d \times (H_m^- \cap Y)$ with the affine space \mathbb{A}^{n-1} , whose coordinates will be denoted by $(a_j)_{j \in J \cup I^\circ}$.

7.4.5 An inductive setup

For each $1 \leq k \leq n-d$, put $I_k^\circ = \{j_k, j_{k+1}, \dots, j_{n-d-1}\}$ and

$$J_k = \{m, 2m, \dots, (n_1-1)m\} \cup J_k^\circ,$$

where

$$J_k^\circ = \begin{cases} \{n - j_k, \dots, n - j_{n-d-1}\} & \text{if } m \text{ is odd or } n_1 \text{ is even,} \\ \{n - j_k, \dots, n - j_{n-d-1}, n/2\} & \text{if } m \text{ is even and } n_1 \text{ is odd.} \end{cases}$$

In particular, $I_1^\circ = I^\circ$, $J_1^\circ = J^\circ$, $I_{n-d}^\circ = \emptyset$, $J_{n-d}^\circ = \emptyset$ if m is odd or n_1 is even, and $J_{n-d}^\circ = \{n/2\}$ if m is even and n_1 is odd. Observe also that for each $1 \leq k \leq n - d - 1$, the set I_{k+1}° (respectively, J_{k+1}) is obtained from I_k° (respectively, J_k) by removing j_k (respectively, $n - j_k$).

We write \mathbb{A}^{n-2k+1} for the $(n - 2k + 1)$ -dimensional affine space, whose coordinates will be denoted by $(a_j)_{j \in J_k \cup I_k^\circ}$. If $1 \leq k \leq n - d - 1$, we write $p_k : \mathbb{A}^{n-2k+1} \rightarrow \mathbb{A}^{n-2k-1}$ for the projection obtained by discarding a_{j_k} and a_{n-j_k} , and $\iota_k : \mathbb{A}^{n-2k-1} \hookrightarrow \mathbb{A}^{n-2k+1}$ for the natural “zero section” of p_k . We put $\alpha_k = \alpha \circ \iota_1 \circ \dots \circ \iota_{k-1} : \mathbb{A}^{n-2k+1} \rightarrow \mathbb{G}_a$, where $\alpha : \mathbb{A}^{n-1} = \mathbb{A}^d \times (H_m^- \cap Y) \rightarrow \mathbb{G}_a$ is the morphism introduced in §7.4.3. In particular, we have $\alpha_1 = \alpha$.

7.4.6 The key lemma

The next result allows us to exploit the inductive setup formulated in §7.4.5.

Lemma 7.9. *For each $1 \leq k \leq n - d - 1$, we have*

$$H_c^\bullet(\mathbb{A}^{n-2k+1}, \alpha_k^* \mathcal{L}_\psi) \cong H_c^\bullet(\mathbb{A}^{n-2k-1}, \alpha_{k+1}^* \mathcal{L}_\psi)[-2](-1)$$

as graded vector spaces with an action of Fr_Q .

Proof. Let \mathbb{A}^{n-2k} be the affine space with coordinates $(a_j)_{j \in J_k \cup I_{k+1}^\circ}$. We factor $p_k : \mathbb{A}^{n-2k+1} \rightarrow \mathbb{A}^{n-2k-1}$ as the composition $p_k = p_k'' \circ p_k'$ of the natural projections

$$\mathbb{A}^{n-2k+1} \xrightarrow{p_k'} \mathbb{A}^{n-2k} \xrightarrow{p_k''} \mathbb{A}^{n-2k-1},$$

and we factor $\iota_k : \mathbb{A}^{n-2k-1} \rightarrow \mathbb{A}^{n-2k+1}$ as the composition $\iota_k = \iota_k'' \circ \iota_k'$ of the natural inclusions

$$\mathbb{A}^{n-2k-1} \xrightarrow{\iota_k'} \mathbb{A}^{n-2k} \xrightarrow{\iota_k''} \mathbb{A}^{n-2k+1}.$$

The assertion of the lemma will follow if we prove that

$$Rp'_{k!}(\alpha_k^* \mathcal{L}_\psi) \cong \iota'_{k!}(\alpha_{k+1}^* \mathcal{L}_\psi)[-2](-1). \quad (7.9)$$

To this end, we will use the proper base change theorem together with the projection formula. We first observe that there exists a polynomial map $\beta_k : \mathbb{A}^{n-2k} \rightarrow \mathbb{G}_a$ such that

$$\begin{aligned} \alpha_k((a_j)_{j \in J_k \cup I_k^\circ}) &= a_{j_k} \cdot a_{n-j_k}^{q^{j_k}} - a_{j_k}^{q^{n-j_k}} \cdot a_{n-j_k}^{q^n} \\ &\quad + \alpha_{k+1}((a_j)_{j \in J_{k+1} \cup I_{k+1}^\circ}) \\ &\quad + a_{n-j_k} \cdot \beta_k((a_j)_{j \in J_k \cup I_{k+1}^\circ}). \end{aligned} \quad (7.10)$$

Indeed, recall that

$$\begin{aligned} \alpha_k((a_j)_{j \in J_k \cup I_k^\circ}) &= \\ &= -p_n \left(\left(1 + \sum_{j \in J_k} a_j^Q \tau^j \right) \cdot \left(1 + \sum_{i \in I_k^\circ} a_i \tau^i \right) \cdot \left(1 + \sum_{j \in J_k} a_j \tau^j \right)^{-1} \right). \end{aligned}$$

The right hand side can be written as a certain sum of monomials in the variables $(a_j)_{j \in J_k \cup I_k^\circ}$. By our choice of the ordering $j_1 > j_2 > \dots > j_{n-d-1} > n/2$, only two of these monomials involve the variable a_{j_k} , namely, $-a_{n-j_k}^Q \cdot a_{j_k}^{q^{n-j_k}}$ and $a_{j_k} \cdot a_{n-j_k}^{q^{j_k}}$. This already implies that we can write

$$\begin{aligned} \alpha_k((a_j)_{j \in J_k \cup I_k^\circ}) &= a_{j_k} \cdot a_{n-j_k}^{q^{j_k}} - a_{j_k}^{q^{n-j_k}} \cdot a_{n-j_k}^{q^n} \\ &\quad + \tilde{\alpha}_{k+1}((a_j)_{j \in J_{k+1} \cup I_{k+1}^\circ}) \\ &\quad + a_{n-j_k} \cdot \beta_k((a_j)_{j \in J_k \cup I_{k+1}^\circ}) \end{aligned}$$

for suitable polynomial maps $\tilde{\alpha}_{k+1} : \mathbb{A}^{n-2k-1} \rightarrow \mathbb{G}_a$ and $\beta_k : \mathbb{A}^{n-2k} \rightarrow \mathbb{G}_a$. Substituting $a_{j_k} = a_{n-j_k} = 0$ into the last identity shows that $\tilde{\alpha}_{k+1} = \alpha_{k+1}$ and yields (7.10).

Next, write $\gamma_k : \mathbb{A}^{n-2k+1} \rightarrow \mathbb{G}_a$ for the morphism

$$(a_j)_{j \in J_k \cup I_k^\circ} \mapsto a_{j_k} \cdot a_{n-j_k}^{q^{j_k}} - a_{j_k}^{q^{n-j_k}} \cdot a_{n-j_k}^{q^n}$$

and define $\delta_k : \mathbb{A}^{n-2k} \rightarrow \mathbb{G}_a$ by

$$(a_j)_{j \in J_k \cup I_{k+1}^\circ} \mapsto \alpha_{k+1}((a_j)_{j \in J_{k+1} \cup I_{k+1}^\circ}) + a_{n-j_k} \cdot \beta_k((a_j)_{j \in J_k \cup I_{k+1}^\circ}).$$

Then (7.10) can be rewritten as $\alpha_k = \gamma_k + \delta_k \circ p'_k$. Since the local system \mathcal{L}_ψ on \mathbb{G}_a is multiplicative, we obtain $\alpha_k^*(\mathcal{L}_\psi) \cong \gamma_k^*(\mathcal{L}_\psi) \otimes p_k'^* \delta_k^*(\mathcal{L}_\psi)$. By the projection formula,

$$Rp_{k!}'(\alpha_k^*(\mathcal{L}_\psi) \cong \delta_k^*(\mathcal{L}_\psi) \otimes Rp_{k!}'(\gamma_k^*(\mathcal{L}_\psi)). \quad (7.11)$$

To analyze $Rp'_{k!}(\gamma_k^* \mathcal{L}_\psi)$, we use the proper base change theorem. The stalk of this complex over a point $x = (a_j)_{j \in J_k \cup I_{k+1}^\circ} \in \mathbb{A}^{n-2k}(\overline{\mathbf{F}}_Q)$ is isomorphic to the compactly supported cohomology on \mathbb{G}_a of the pullback of \mathcal{L}_ψ via the map

$$f_{a_{n-j_k}} : \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad y \longmapsto a_{n-j_k}^{q^{j_k}} \cdot y - a_{n-j_k}^{q^n} \cdot y^{q^{n-j_k}}.$$

Using the notation of §5.4, we have, by Corollary 5.10,

$$f_{a_{n-j_k}}^*(y) = a_{n-j_k}^{q^{j_k}} \cdot y - a_{n-j_k}^{q^n/q^{n-j_k}} \cdot y^{1/q^{n-j_k}} = a_{n-j_k}^{q^{j_k}} \cdot (y - y^{q^{j_k-n}}).$$

So if $a_{n-j_k} \neq 0$, then $f_{a_{n-j_k}}^*(y)$ can only be zero when $y \in \mathbf{F}_{q^{n-j_k}}$. Since $m \nmid (n-j_k)$ by construction and the character ψ has conductor q^m , we see that $f_{a_{n-j_k}}^*(\mathcal{L}_\psi)$ is trivial if and only if $a_{n-j_k} = 0$.

Lemma 5.7 and the proper base change theorem imply that the complex $Rp'_{k!}(\gamma_k^* \mathcal{L}_\psi)$ vanishes away from the image of $\iota'_k : \mathbb{A}^{n-2k-1} \hookrightarrow \mathbb{A}^{n-2k}$. On the other hand, a calculation similar to the one done in the previous paragraph shows that the restriction of $\gamma_k^* \mathcal{L}_\psi$ to

$$p'^{-1}(\iota'_k(\mathbb{A}^{n-2k-1})) \subset \mathbb{A}^{n-2k+1}$$

is trivial. Since $p'^{-1}(\iota'_k(\mathbb{A}^{n-2k-1}))$ can be identified with the product $\iota'_k(\mathbb{A}^{n-2k-1}) \times \mathbb{A}^1$ in such a way that p'_k becomes the first projection, the proper base change theorem implies that

$$Rp'_{k!}(\gamma_k^* \mathcal{L}_\psi) \cong \iota'_{k!} \overline{\mathbb{Q}}_\ell[-2](-1),$$

where $\overline{\mathbb{Q}}_\ell$ stands for the constant sheaf on \mathbb{A}^{n-2k-1} . Therefore, by (7.11),

$$Rp'_{k!}(\alpha_k^* \mathcal{L}_\psi) \cong \iota'_{k!}(\iota'_k{}^* \delta_k^*(\mathcal{L}_\psi))[-2](-1).$$

Finally, by construction, $\delta_k \circ \iota'_k = \alpha_{k+1}$, which yields (7.9) and finishes the proof of Lemma 7.9. \square

7.4.7 Conclusion of the proof of Proposition 7.5

We return to the task of proving (7.6)–(7.8). Applying Lemma 7.9 successively for $k = 1, 2, \dots, n-d-1$, we obtain

$$\begin{aligned} H_c^\bullet(\mathbb{A}^d \times (H_m^- \cap Y), \alpha^* \mathcal{L}_\psi) &= H_c^\bullet(\mathbb{A}^{n-1}, \alpha^* \mathcal{L}_\psi) \\ &\cong H_c^\bullet(\mathbb{A}^{n-2(n-d)+1}, \alpha_{n-d}^* \mathcal{L}_\psi)[-2(n-d-1)](-(n-d-1)). \end{aligned} \quad (7.12)$$

Let us now calculate $\alpha_{n-d}^*(\mathcal{L}_\psi)$. Recall that the coordinates on the affine space $\mathbb{A}^{n-2(n-d)+1}$ are labeled $(a_j)_{j \in J_{n-d}}$, where

$$J_{n-d} = \begin{cases} \{m, 2m, \dots, (n_1 - 1)m\} & \text{if } m \text{ is odd or } n_1 \text{ is even,} \\ \{m, 2m, \dots, (n_1 - 1)m, n/2\} & \text{if } m \text{ is even and } n_1 \text{ is odd,} \end{cases}$$

and

$$\alpha_{n-d}((a_j)_{j \in J_{n-d}}) = -\text{pr}_n \left(\left(1 + \sum_{j \in J_{n-d}} a_j^Q \tau^j \right) \cdot \left(1 + \sum_{j \in J_{n-d}} a_j \tau^j \right)^{-1} \right).$$

So if m is odd or n_1 is even, we can naturally identify $\mathbb{A}^{n-2(n-d)+1}$ with

$$(U^{n_1, q_1} \cap Y) \subset U^{n_1, q_1} \subset U^{n, q},$$

and α_{n-d} with the map

$$U^{n_1, q_1} \cap Y \longrightarrow \mathbb{G}_a, \quad g \longmapsto -\text{pr}_n(L_Q(g)).$$

Recall that $Q = q^n = q_1^{n_1}$. Applying Proposition 6.2 with (n, q) replaced by (n_1, q_1) , we find that $-\text{pr}_n(L_Q(g)) = N^{n_1, q_1}(g) - N^{n_1, q_1}(g)^{q_1}$ for all $g \in U^{n_1, q_1}$. Since $q_1 = q^m$ is the conductor of ψ , the pullback of \mathcal{L}_ψ by the map $x \mapsto x - x^{q_1}$ is trivial. Therefore $\alpha_{n-d}^*(\mathcal{L}_\psi)$ is the trivial local system on $\mathbb{A}^{n-2(n-d)+1}$, whence

$$H_c^\bullet(\mathbb{A}^{n-2(n-d)+1}, \alpha_{n-d}^* \mathcal{L}_\psi) \cong \overline{\mathbb{Q}}_\ell[-2(n-2(n-d)+1)](-(n-2(n-d)+1)).$$

Combining this with (7.12) yields

$$H_c^\bullet(\mathbb{A}^d \times (H_m^- \cap Y), \alpha^* \mathcal{L}_\psi) \cong \overline{\mathbb{Q}}_\ell[-2d](-d).$$

Recalling that $d = (n + n_1 - 2)/2$ when m is odd or n_1 is even, we obtain all the desired assertions (7.6)–(7.8) in this case.

Suppose next that m is even and n_1 is odd. Then we can naturally identify $\mathbb{A}^{n-2(n-d)+1}$ with $(U^{n_1, q_1} \cap Y) \times \mathbb{G}_a$, where the first factor $U^{n_1, q_1} \cap Y$ corresponds to the coordinates $(a_j)_{j=m, 2m, \dots, (n_1-1)m}$ and the second factor \mathbb{G}_a corresponds to the coordinate $a_{n/2}$. Under this identification, α_{n-d} corresponds to the map

$$(g, x) \longmapsto -\text{pr}_n(L_Q(g)) + x^{q^n + q^{n/2}} - x^{1 + q^{n/2}}.$$

As in the previous case, the pullback of \mathcal{L}_ψ by the map $(g, x) \mapsto -\text{pr}_n(L_Q(g))$ is trivial. Since the local system \mathcal{L}_ψ is multiplicative, we see that $H_c^\bullet(\mathbb{A}^{n-2(n-d)+1}, \alpha_{n-d}^* \mathcal{L}_\psi)$ is isomorphic to

$$H_c^\bullet(\mathbb{G}_a, f^*(\mathcal{L}_\psi))[-2(n-2(n-d))(-(n-2(n-d))),$$

where $f : \mathbb{G}_a \rightarrow \mathbb{G}_a$ is given by $x \mapsto x^{q^n+q^{n/2}} - x^{1+q^{n/2}}$. So by (7.12),

$$\begin{aligned} & H_c^\bullet(\mathbb{A}^d \times (H_m^- \cap Y), \alpha^* \mathcal{L}_\psi) \cong \\ & \cong H_c^\bullet(\mathbb{G}_a, f^*(\mathcal{L}_\psi))[-2(d-1)](-(d-1)) \end{aligned} \quad (7.13)$$

Now we can factor f as $f = f_1 \circ f_2$, where $f_1(x) = x^{q^{n/2}} - x$ and $f_2(x) = x^{1+q^{n/2}}$. Thus $f^*(\mathcal{L}_\psi) \cong f_2^* f_1^*(\mathcal{L}_\psi)$. Since f_1 is a homomorphism, $f_1^*(\mathcal{L}_\psi) \cong \mathcal{L}_{\psi \circ f_1}$ is the multiplicative local system on \mathbb{G}_a corresponding to the character $\psi \circ f_1 : \mathbf{F}_Q \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Now since ψ has conductor q^m and $m \nmid (n/2)$ by assumption, $\psi \circ f_1$ is nontrivial. On the other hand, $(\psi \circ f_1)|_{\mathbf{F}_{q^{n/2}}}$ is necessarily trivial. Applying Proposition 7.10 below to $q^{n/2}$ in place of q and $\psi \circ f_1$ in place of ψ , we see that

$$\dim H_c^r(\mathbb{G}_a, f_2^* \mathcal{L}_{\psi \circ f_1}) = \begin{cases} q^{n/2} & \text{if } r = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (7.14)$$

and

$$\text{Fr}_Q \text{ acts on } H_c^1(\mathbb{G}_a, f_2^* \mathcal{L}_{\psi \circ f_1}) \text{ via } -q^{n/2} \quad (7.15)$$

Recalling that $d = (n + n_1 - 1)/2$ in the situation under consideration, we see that all the desired assertions (7.6)–(7.8) follow in this case from (7.13), (7.14) and (7.15).

7.5 An auxiliary cohomology calculation

The next result was used in §7.4.

Proposition 7.10. *Let $\psi : \mathbf{F}_{q^2} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a nontrivial character such that ψ is trivial on $\mathbf{F}_q \subset \mathbf{F}_{q^2}$, and let \mathcal{L}_ψ be the corresponding⁷ multiplicative local system on \mathbb{G}_a over \mathbf{F}_{q^2} . Write $f : \mathbb{G}_a \rightarrow \mathbb{G}_a$ for the map $x \mapsto x^{q+1}$. Then*

$$\dim H_c^i(\mathbb{G}_a, f^* \mathcal{L}_\psi) = \begin{cases} q & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

⁷In the sense of Lemma 5.6.

Moreover, the Frobenius Fr_{q^2} acts on $H_c^1(\mathbb{G}_a, f^*\mathcal{L}_\psi)$ via multiplication by $-q$.

Proof. Since $x^{q+1} \in \mathbf{F}_q$ for all $x \in \mathbf{F}_{q^2}$, we have the identity

$$\sum_{x \in \mathbf{F}_{q^2}} \psi(x^{q+1}) = q^2, \quad (7.16)$$

which is consistent with the assertion of the proposition in view of the Grothendieck-Lefschetz trace formula. We will deduce the proposition from (7.16). To this end, observe that if $\text{pr} : \mathbb{G}_a \rightarrow \text{Spec } \mathbf{F}_{q^2}$ denotes the structure morphism, then

$$R\text{pr}_!(f^*\mathcal{L}_\psi) \cong R\text{pr}_!(f_!f^*\mathcal{L}_\psi) \cong R\text{pr}_!(\mathcal{L}_\psi \otimes f_!\overline{\mathbb{Q}}_\ell), \quad (7.17)$$

where $\overline{\mathbb{Q}}_\ell$ is the constant rank 1 local system on \mathbb{G}_a and in the second isomorphism we used the projection formula.

Lemma 7.11. *One has*

$$f_!\overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus \bigoplus_{s=1}^q j_!(\mathcal{M}_s)$$

for certain nontrivial multiplicative local systems $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_q$ on \mathbb{G}_m over \mathbf{F}_{q^2} , where $j : \mathbb{G}_m \hookrightarrow \mathbb{G}_a$ denotes the inclusion map.

Before proving this lemma, let us explain why it implies the assertion of the proposition. By the lemma and (7.17), we have

$$R\text{pr}_!(f^*\mathcal{L}_\psi) \cong R\text{pr}_!(\mathcal{L}_\psi) \oplus \bigoplus_{s=1}^q R\text{pr}_!(\mathcal{L}_\psi \otimes j_!\mathcal{M}_s). \quad (7.18)$$

Now $R\text{pr}_!(\mathcal{L}_\psi) = 0$ by Lemma 5.7. By [Del77, Prop. 4.2 in *Sommes Trig.*],

$$\dim H_c^r(\mathbb{G}_a, \mathcal{L}_\psi \otimes j_!\mathcal{M}_s) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and if λ_s is the scalar by which Fr_{q^2} acts on $H_c^1(\mathbb{G}_a, \mathcal{L}_\psi \otimes j_!\mathcal{M}_s)$, then $|\lambda_s| = q$. Applying the Grothendieck-Lefschetz trace formula to (7.18) yields

$$\sum_{x \in \mathbf{F}_{q^2}} \psi(x^{q+1}) = -(\lambda_1 + \dots + \lambda_q).$$

Comparing this with (7.16) and using the fact that $|\lambda_s| = q$ for all s , we see that $\lambda_s = -q$ for all s , which, in view of (7.18), yields the proposition. \square

Proof of Lemma 7.11. Since $f^{-1}(0) = \{0\}$, the stalk of $f_!(\overline{\mathbb{Q}}_\ell)$ at 0 is 1-dimensional. Since f is a finite morphism, we have $f_!(\overline{\mathbb{Q}}_\ell) = f_*(\overline{\mathbb{Q}}_\ell)$, and by adjunction, there is a natural map $\overline{\mathbb{Q}}_\ell \rightarrow f_*(\overline{\mathbb{Q}}_\ell)$ (coming from the natural isomorphism $f^*(\overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$), which induces an isomorphism on the stalks over 0.

Next let us calculate the restriction of $f_!(\overline{\mathbb{Q}}_\ell)$ to $\mathbb{G}_m \subset \mathbb{G}_a$. To this end, consider the restriction of f to a morphism $\mathbb{G}_m \rightarrow \mathbb{G}_m$. Since $f'(x) = (q+1)x^q = x^q$, the map $f : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is étale, and in fact, it can be identified with the quotient of \mathbb{G}_m by the finite discrete⁸ subgroup $\mu_{q+1}(\mathbf{F}_{q^2}) \subset \mathbb{G}_m$ of $(q+1)$ -st roots of unity. This means that the restriction $f_!(\overline{\mathbb{Q}}_\ell)|_{\mathbb{G}_m}$ decomposes as a direct sum $\overline{\mathbb{Q}}_\ell \oplus \bigoplus_{s=1}^q \mathcal{M}_s$, where the \mathcal{M}_s are the local systems coming from the nontrivial characters of $\mu_{q+1}(\mathbf{F}_{q^2})$.

In particular, for each s , we have a map $\mathcal{M}_s \hookrightarrow f_!(\overline{\mathbb{Q}}_\ell)|_{\mathbb{G}_m}$, which by adjunction induces a map $j_!\mathcal{M}_s \rightarrow f_!(\overline{\mathbb{Q}}_\ell)$. Finally, combining these maps with the map $\overline{\mathbb{Q}}_\ell \rightarrow f_!(\overline{\mathbb{Q}}_\ell)$ constructed in the first paragraph of the proof, we obtain a map

$$\overline{\mathbb{Q}}_\ell \oplus \bigoplus_{s=1}^q j_!(\mathcal{M}_s) \rightarrow f_!(\overline{\mathbb{Q}}_\ell).$$

By looking at the stalks, one sees that this map is an isomorphism. \square

7.6 Proof of Proposition 7.6

As in §7.4, all varieties and algebraic groups we will consider are assumed to be defined over \mathbf{F}_Q . We recall that $\tilde{\psi} : H_m(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is the character defined by $\tilde{\psi} = \psi_1 \circ \text{Nm}^{n_1, q_1} \circ \nu_m$ (see §7.1).

In view of the last statements of parts (a) and (b) of Proposition 7.4, the assertion of Proposition 7.6 is equivalent to the following claim:

$$\text{Hom}_{U^{n,q}(\mathbf{F}_Q)} \left(\text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)} (\tilde{\psi}), H_c^\bullet(X, \overline{\mathbb{Q}}_\ell) \right) \neq 0.$$

Let $\mathcal{E}_{\tilde{\psi}}$ be the local system on H_m coming from the Lang isogeny $L_Q : H_m \rightarrow H_m$ via $\tilde{\psi}$. We simply write \mathcal{E} for the local system on $U^{n,q}$ coming from the Lang isogeny $L_Q : U^{n,q} \rightarrow U^{n,q}$ via $\text{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)} (\tilde{\psi})$.

⁸Observe that all the $(q+1)$ -st roots of unity already lie in \mathbf{F}_{q^2} .

By Corollary 5.13, we have

$$\mathrm{Hom}_{U^{n,q}(\mathbf{F}_Q)}\left(\mathrm{Ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\tilde{\psi}), H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)\right) \cong H_c^\bullet(Y, \mathcal{E}|_Y).$$

So in order to show that the left hand side is nonzero, it suffices (by the Grothendieck-Lefschetz trace formula) to check that

$$\sum_{y \in Y(\mathbf{F}_Q)} t_{\mathcal{E}}(y) \neq 0. \quad (7.19)$$

Now by Corollary 5.17,

$$t_{\mathcal{E}} = \mathrm{ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(t_{\mathcal{E}_{\tilde{\psi}}}). \quad (7.20)$$

To compute the right hand side of the last identity, we use

Lemma 7.12. *We have⁹ $t_{\mathcal{E}_{\tilde{\psi}}} = \psi \circ \mathrm{pr}_n : H_m(\mathbf{F}_Q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$.*

Proof. The projection $\nu_m : H_m \longrightarrow U^{n_1, q_1}$ obtained by discarding all summands $a_j \tau^j$ with $m \nmid j$ is an algebraic group homomorphism (cf. Remark 4.3), whose kernel is equal to $H_m^- \cap Y$. Moreover, if we view U^{n_1, q_1} as a subgroup of $U^{n, q}$ as explained earlier, then $U^{n_1, q_1} \subset H_m^-$ and ν_m restricts to the identity on U^{n_1, q_1} . So we obtain a semidirect product decomposition $H_m = U^{n_1, q_1} \ltimes (H_m^- \cap Y)$.

Now to calculate the function $t_{\mathcal{E}_{\tilde{\psi}}} : H_m(\mathbf{F}_Q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ we use Proposition 5.15. Fix $\gamma \in H_m(\mathbf{F}_Q)$ and choose $g \in H_m(\overline{\mathbf{F}}_Q)$ with $L_Q(g) = \gamma$. Write $g = g_1 \cdot h$ for uniquely determined $g_1 \in U^{n_1, q_1}(\overline{\mathbf{F}}_Q)$ and $h \in (H_m^- \cap Y)(\overline{\mathbf{F}}_Q)$. Then

$$\nu_m(g^{-1} \cdot \mathrm{Fr}_Q(g)) = \nu_m(h^{-1} \cdot g_1^{-1} \cdot \mathrm{Fr}_Q(g_1) \cdot \mathrm{Fr}_Q(h)) = g_1^{-1} \cdot \mathrm{Fr}_Q(g_1)$$

because ν_m is an algebraic group homomorphism, so by Proposition 5.15,

$$t_{\mathcal{E}_{\tilde{\psi}}}(\gamma) = \tilde{\psi}(g^{-1} \cdot \mathrm{Fr}_Q(g)) = \psi_1(\mathrm{Nm}^{n_1, q_1}(g_1^{-1} \cdot \mathrm{Fr}_Q(g_1))).$$

Now $g_1^{-1} \cdot \mathrm{Fr}_Q(g_1) \in U^{n_1, q_1}(\mathbf{F}_Q)$ and $\mathrm{Fr}_Q(g_1) = g_1 \cdot (g_1^{-1} \cdot \mathrm{Fr}_Q(g_1))$. Applying Corollary 6.4 and Lemma 6.1 to the reduced norm map $N^{n_1, q_1} : U^{n_1, q_1} \longrightarrow \mathbb{G}_a$, we obtain

$$\begin{aligned} \mathrm{Nm}^{n_1, q_1}(g_1^{-1} \cdot \mathrm{Fr}_Q(g_1)) &= N^{n_1, q_1}(\mathrm{Fr}_Q(g_1)) - N^{n_1, q_1}(g_1) \\ &= \mathrm{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q_1}}(\mathrm{pr}_n(L_Q(g_1))), \end{aligned}$$

⁹We point out that in general, $\tilde{\psi} \neq \psi \circ \mathrm{pr}_n$ on $H_m(\mathbf{F}_Q)$ and $\psi \circ \mathrm{pr}_n : H_m(\mathbf{F}_Q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is not a group homomorphism.

where in the last step we used Corollary 6.5. Here we note that $L_Q(g_1) \in U^{n_1, q_1}(\mathbf{F}_Q)$ because $L_Q(g_1) = \nu_m(\gamma)$. Now $\text{pr}_n(L_Q(g_1)) = \text{pr}_n(\gamma)$, so we finally obtain

$$t_{\mathcal{E}_{\bar{\psi}}}(\gamma) = \psi_1(\text{Tr}_{\mathbf{F}_Q/\mathbf{F}_{q_1}}(\text{pr}_n(L_Q(g_1)))) = \psi(\text{pr}_n(\gamma)),$$

which completes the proof of Lemma 7.12. \square

Let us now verify (7.19). By (7.20) and Lemma 7.12,

$$t_{\mathcal{E}} = \text{ind}_{H_m(\mathbf{F}_Q)}^{U^{n,q}(\mathbf{F}_Q)}(\psi \circ \text{pr}_n),$$

so if $\{g_i\}_{i \in I}$ are representatives of all the left cosets of $H_m(\mathbf{F}_Q)$ in $U^{n,q}(\mathbf{F}_Q)$, then

$$\sum_{y \in Y(\mathbf{F}_Q)} t_{\mathcal{E}}(y) = \sum_{i \in I} \sum_{y \in Y(\mathbf{F}_Q) \cap (g_i \cdot H_m(\mathbf{F}_Q) \cdot g_i^{-1})} \psi(\text{pr}_n(g_i^{-1} y g_i)). \quad (7.21)$$

We will show that the right hand side is a strictly positive integer. To this end, consider a new group operation on $U^{n,q}$, which we denote by \boxplus and define by

$$\left(1 + \sum_{j=1}^n a_j \tau^j\right) \boxplus \left(1 + \sum_{j=1}^n b_j \tau^j\right) = 1 + \sum_{j=1}^n (a_j + b_j) \tau^j.$$

For any $g \in U^{n,q}$, the map $x \mapsto gxg^{-1}$ is a homomorphism with respect to \boxplus (where we denote the old group operation on $U^{n,q}$ multiplicatively, as usual), as is the map $\text{pr}_n : U^{n,q} \longrightarrow \mathbb{G}_a$. Hence for each $i \in I$, the subset

$$Y(\mathbf{F}_Q) \cap (g_i \cdot H_m(\mathbf{F}_Q) \cdot g_i^{-1}) \subset U^{n,q}(\mathbf{F}_Q)$$

is a subgroup with respect to \boxplus , and the map $y \mapsto \psi(\text{pr}_n(g_i^{-1} y g_i))$ is a character of this subgroup. Therefore the i -th summand,

$$\sum_{y \in Y(\mathbf{F}_Q) \cap (g_i \cdot H_m(\mathbf{F}_Q) \cdot g_i^{-1})} \psi(\text{pr}_n(g_i^{-1} y g_i)),$$

is equal to either 0 or the order of $Y(\mathbf{F}_Q) \cap (g_i \cdot H_m(\mathbf{F}_Q) \cdot g_i^{-1})$, which is a positive integer. Finally note that there is an $i_0 \in I$ for which $g_{i_0} \in H_m(\mathbf{F}_Q)$. Then the corresponding character $y \mapsto \psi(\text{pr}_n(g_{i_0}^{-1} y g_{i_0}))$ of $Y(\mathbf{F}_Q) \cap (g_{i_0} \cdot H_m(\mathbf{F}_Q) \cdot g_{i_0}^{-1}) = Y(\mathbf{F}_Q) \cap H_m(\mathbf{F}_Q)$ is equal to $y \mapsto \psi(\text{pr}_n(y)) \equiv \psi(0) = 1$ because $Y = \text{pr}_n^{-1}(0)$. So the summand corresponding to $i = i_0$ in (7.21) is positive, which yields (7.19).

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